République Algérienne Démocratique
et Populaire

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Hadj Lakhdar de Batna

Faculté des Sciences de l'ingénieur

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en vue de l'obtention du grade de
Docteur en Science

Non Linear Systems Control: LMI Fuzzy Approach

Soutenue le: 06/07/2009

Devant le jury

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Année Universitaire 2008/2009

Dedication

To my dear Parents

Acknowledgments

First of all, I would like to express my sincere gratitude to my advisor, Prof. Noureddine Goléa, for his solid technical guidance, continuous support and encouragement in my work and for his responsiveness to my requests.

I am also grateful to Dr. Lamir Saidi, Dr. Kamel kara and Dr. Mohamed Boumehrez, who accepted to be my jury members, and devoted their precious time to review my thesis. I also want to thank Dr. Ahmed Louchene for having agreed to preside this jury.

I would like to thank Dr. Mohammed Laid Hadjili from Belgium, for his orientations and his judicious councils at the beginning of my research.

I also thanks Pr. Ayache Bouakaz to have received me within his laboratory at François Rabelais University in Tours, France, to advance in my research.

Finally, my deep recognition goes to my dear parents Abdelhamid, Farida and yemma Fatima, to my parents in law Ali and Messaouda, for all their encouragements and continuous support during this period of research. I am also grateful to all my family: my children Billel and Tasnim, my niece Mourdjane, my dear sister Ahlem, my brothers Med El Hachemi and Nidhal, my aunt Zazi and my uncle Abdelaziz. A particular grateful is for my husband Dr. Nabil Benoudjit, who helped me, by his councils, his moral support and his encouragements, with a great patience. Once again thanks for all.

Abstract

The objective of this thesis is the development of new Lyapunov stability conditions for continuous Takagi-Sugeno fuzzy systems, in order to reduce the degree of conservatism. The nonlinear systems are represented and controlled by Takagi-Sugeno fuzzy model design. This design combines the flexibility of fuzzy logic theory and the rigorous mathematical analysis tools in linear system theory in a unified framework. Takagi-Sugeno fuzzy systems allow a multimodel representation, which is a convex polytopic form. The most used fuzzy control design in the literature is carried out using the Parallel Distributed Compensation (PDC) scheme since it shares the same membership functions of the T-S fuzzy model. The main idea of the PDC controller design is to derive each control rule from the corresponding rule of T-S fuzzy model so as to compensate it. The resulting overall fuzzy controller, which is nonlinear in general, is a fuzzy blending of each individual state feedback linear controller. The advantage of the T-S fuzzy model lies in that the stability and performance characteristics of the system represented by a T-S fuzzy model can be analyzed using Lyapunov function approach where stability conditions resolution depends on a set of Linear Matrix Inequalities (LMIs).

In this thesis, new non-quadratic stability conditions are derived based on Parallel Distributed Compensation (PDC) to stabilize continuous T-S fuzzy systems and on fuzzy Lyapunov functions. We obtain new conditions, shown to be less conservative, that stabilize continuous T-S fuzzy systems including those that do not admit a quadratic stabilization. Our approach is based on two assumptions. The first one relies on the existence of a proportionality relation between multiple Lyapunov functions, and the second one considers an upper bound for the time derivative of the premise membership function. The obtained stability results are extended to the case where the states are not available for measurement and feedback by using fuzzy observer, while guarantying the stability of the whole system. Whereas, to check the stability of the whole system i.e. (fuzzy system+fuzzy controller+fuzzy observer), we applied a separation property. Different examples are presented to show the effectiveness of our proposal.

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Chapter 1

Introduction

1.1 Context and motivations

During the last decade, fuzzy logic has attracted great attention, because of its ability to simultaneously handle numerical data and linguistic data knowledge. Fuzzy sets theory was first introduced in the landmark paper of Zadeh (Zadeh, 1965) at berekly university. Fuzzy logic is a powerful problem-solving methodology with a myriad of applications embedded in control and information processing since 70's years, such as Mamdani which concretizes the first time this method to realize a fuzzy control in an industrial application (Mamdani, 1974),(Mamdani, 1977). Hence, unlike classical logic, which requires a deep understanding of a system, exact equations and precise numeric values, fuzzy logic incorporates an alternative way of thinking, which allows modeling complex systems using a higher level of abstraction originating from our knowledge and experience. The human expertise is used to construct a set of fuzzy rules of the form "IF X is A THEN Y is B", allowing the construction of fuzzy models, especially for systems that are difficult to modelize, and consequently the number of applications based on fuzzy logic increased these last years considerably such as modelization, control, signal processing, pattern recognition and expert systems fields. The principal advantage of fuzzy logic systems is their aptitude to approximate any nonlinear function; they are universal approximators (Wang & Mendel, 1992). Many researches have been developed to demonstrate this concept (Kosko, 1994),(Castro, 1995),(Ying, 1998),(Zeng *et al.* , 2000),(Sala & Arino, 2007). The common point of this researches is the capability of a fuzzy model to approximate and then to represent any real function. A theoretical justification of fuzzy models as universal approximators has been given by wang (Wang & Mendel, 1992) for standard fuzzy systems with gaussian membership functions, product implication, conjunction and center of gravity defuzzification.

A fuzzy model is an expression of the system of interest in the framework of a fuzzy logic. There exist two kinds of fuzzy models: Mamdani fuzzy models and Takagi-Sugeno (T-S) fuzzy models. The advantage of T-S fuzzy models is their powerful capability to represent a complex nonlinear relationship in spite of a smaller number of fuzzy IF-THEN rules, than that of the Mamdani model. Moreover, in this type of model, the passage from one rule to another is done by a smooth transition from one rule to another, i.e., an interpolation between the rules.

In the literature, it appears that the most important application of fuzzy logic is fuzzy control, that was developed in Europe from the eighties and whose many researches works were produced such as in japan (Takagi & Sugeno, 1985), in the following decade. Takagi and Sugeno (Takagi & Sugeno, 1985) proposed a multimodel based approach to overcome the difficulties of the conventional modeling techniques. The proposed multimodel (T-S) is a convex polytopic form and can be obtained from identification approach (Sugeno & Kang, 1995),(Babuska, 1998) or from nonlinear dynamical model, by linearization, by the principle of sector nonlinearity (Kawamoto *et al.* , 1992) or by local approximation (Tanaka & Wang, 2001, a). For this purpose, a nonlinear plant is represented by the T-S fuzzy model, where local dynamics in different state regions are represented by linear models. The overall model of the system is obtained by a fuzzy blending of these local models. This same fuzzy structure is used to control (Takagi & Sugeno, 1985),(Tanaka & Sugeno, 1992) (Wang *et al.* , 1996),(Feng, 2002) and to study the stability of the T-S fuzzy system using Lyapunov method (Tanaka & Sugeno, 1992),(Zhao, 1995) and Linear Matrix Inequalities (LMI), where the problem can be numerically solved by convex optimization techniques (Tanaka & Sugeno, 1992),(Boyd *et al.* , 1994),(Tanaka *et al.* , 2001, b).

The most used fuzzy control design in the literature is carried out using the Parallel Distributed Compensation (PDC) scheme (Tanaka & Sugeno, 1992),(Wang *et al.* , 1996) since it shares the same membership functions of the T-S fuzzy model. The main idea of the PDC controller design is to derive each linear control rule from the corresponding rule of T-S fuzzy model so as to compensate it. The resulting overall fuzzy controller, which is nonlinear in general, is a fuzzy blending of the local linear controllers, knowing that the fuzzy controller shares the same fuzzy sets with the fuzzy model. Wang et al. (Wang *et al.* , 1996) used this concept to design fuzzy controllers to stabilize T-S fuzzy systems.

The advantage of the T-S fuzzy model lies in that the stability and performance characteristics of the system represented by a T-S model can be analyzed using Lyapunov function approach (Tanaka & Sugeno, 1992),(Zhao, 1995). Tanaka and Sugeno (Tanaka & Sugeno, 1992) showed that the stability of a T-S fuzzy model could be shown by finding a common symmetric positive definite matrix P for r sub-models, that satisfy a set of Lyapunov inequalities (Tanaka *et al.* , 1996),(Wang *et al.* , 1996). Hence, stability conditions are derived using a Lyapunov stability criteria for the fuzzy model , leading usually to Linear Matrix Inequalities (LMI) conditions, which are numerically tractable. Different works have been realized based on this approach, such as Lee et al. (Lee *et al.* , 2001) who proposed a robust fuzzy control scheme for nonlinear systems in the presence of parametric uncertainties, where sufficient conditions were derived for robust stabilization in the sense of Lyapunov stability and Cao and Lin (Cao & Lin, 2003) who applied the Lyapunov function based approach for the stability analysis of nonlinear systems with actuator saturation. On the other hand, Tsen et al. (Tsen *et al.* , 2001) proposed a fuzzy H_∞ model reference tracking control scheme and discussed the stability of the closed loop nonlinear system by Lyapunov approach, and Korba et al. (Korba *et al.* , 2003) presented a constructive and automated method for the design of a gain-scheduling controller, based on a given T-S fuzzy model and a controller that guarantees the closed loop stability using Lyapunov quadratic functions. However, a possible limitation of their approaches is the use of the quadratic Lyapunov method, which is conservative. The quadratic approach requires to find a common positive definite matrix P for r sub-models, what makes it very conservative and hence brings us to search for less conservative stability conditions. Thus, having a T-S fuzzy model, the fundamental difficulty which arises during the synthesis of PDC controller is the conservatism of the stability conditions. By consequent, with an aim of having less conservative results , LMI relaxed conditions were the object of several works, in particular those developed in (Tanaka *et al.* , 1998) where the authors base themselves on the maximum number of active rules at each moment to reduce the conservatism of stabilization conditions. Kim and Lee (Kim & Lee, 2000) take as a starting point this work, by introducing additional conditions. In (Cao *et al.* , 1997),(Jadbabaie, 1999),(Chadli *et al.* , 2000),(Tanaka *et al.* , 2001, c),(Hadjili, 2002),(Teixeira *et al.* , 2003), the authors propose to use multiple Lyapunov functions to search for several positive definite matrices instead of searching for a common one, using quadratic Lyapunov function. Whereas, Johansson (Johansson *et al.* , 1999, a),(Rantzer & Johansson, 2000) used piecewise Lyapunov functions to reduce conservatism.

However, the states of a system are not always available for measurement which is the case in a lot of practical problems. To overcome this limit, the notion of observer was introduced. The concept of linear regulator and linear observer were introduced by Kalman (Kalman, 1961) for linear systems in stochastic environment and by Luenberger (Luenberger, 1966) for deterministic linear systems, whereas for nonlinear systems, different observer designs were proposed such as the extended kalman observer, the sliding mode observer (Utkin & Drakunov, 1995), the high gain observer

(Nicosia & Tornambe, 1989) and the T-S fuzzy observer, that was introduced by several authors in the literature such as Tanaka (Tanaka & Sano, 1994), Feng et al. (Feng *et al.* , 1997),(Lee *et al.* , 2001) who proposed fuzzy observers with an asymptotic convergence. Tanaka proposed in his paper (Tanaka *et al.* , 1998) a globally exponentially stable fuzzy controllers and fuzzy observers designs for continuous and discrete fuzzy systems for both measurable and non measurable premise variables. Other approaches were proposed by different authors, among them, Fayaz (Fayaz, 2000) who combined the results of (Tanaka *et al.* , 1998),(Fayaz, 1999) by using local Lyapunov functions to prove the existence of globally and quadratically stabilizing regulator and observer, Ma and Sun (Ma & Sun, 2001) for T-S fuzzy systems analysis and design of reduced-dimensional fuzzy observer and fuzzy functional observer with a separation property, and also (Cao & Frank, 2000),(Chen & Liu, 2004),(Wang, 2004),(Chen & Liu, 2005),(Lin *et al.* , 2006) and (Lin *et al.* , 2008) with other considerations. Hence, the observer design is a very important problem in control systems and the stability of the whole system, with the fuzzy controller and the fuzzy observer, must be guaranteed. For a T-S fuzzy system, a separation property is used to check the stability of the global system. This concept was introduced by Jadbabaie et al. (Jadbabaie, 1997, b) and Ma et al. (Ma *et al.* , 1998) by different approaches to assure an independent design for the controller and the observer while assuring the stability of the global T-S system.

1.2 Objectives and contributions

The objective of this research is the development of new Lyapunov stability conditions for continuous T-S fuzzy systems, in order to reduce the degree of conservatism. Hence, new non-quadratic stability conditions are derived based on PDC to stabilize continuous T-S fuzzy models. We use the fuzzy Lyapunov function since it is smooth contrary to the piecewise Lyapunov function thus avoiding the boundary condition problem. We obtain new conditions, shown to be less conservative, that stabilize continuous fuzzy systems including those that do not admit a quadratic stabilization. Our approach is based on two assumptions. The first one relies on the existence of a proportionality relation between multiple quadratic Lyapunov functions, and the second one considers an upper bound for the time derivative of the premise membership function as assumed by Tanaka et al. (Tanaka *et al.* , 2001, b),(Tanaka *et al.* , 2001, c),(Tanaka *et al.* , 2001, d),(Tanaka *et al.* , 2003). We extend the stability results given in (Abdelmalek *et al.* , 2007) to the case of non available states for measurement and feedback, i.e. to the fuzzy observer conception, while guaranteeing the stability of the whole system. Whereas, we applied the separation principle of Ma et al.(Ma *et al.* , 1998), due to its simplicity, since it does not depend on the stability conditions but rather on the fuzzy Lyapunov functions. Indeed, the separation principle design proposed in (Jadbabaie, 1997, b) is not appropriated for the case of several stability conditions. All these steps are illustrated by different examples.

1.3 Organization of the thesis

This thesis is organized as follows:

Chapter 2 introduces the fuzzy modeling by two particular structures of fuzzy systems that are Mamdani fuzzy systems and T-S fuzzy systems, followed by the different construction methods of T-S fuzzy models due to their interesting characteristics. Different existing methods for constructing a T-S fuzzy model are detailed and illustrated by different examples. This chapter finishes by a theorem on the concept of "fuzzy systems are universal approximators", for any real continuous function.

Chapter 3 is devoted to quadratic stability and stabilization of T-S fuzzy systems by Lyapunov method. First, a recall is given on stability definition in the Lyapunov sense. Then quadratic stability conditions proposed by Tanaka (Tanaka & Sugeno, 1992),(Tanaka & Wang, 2001, a) are given for continuous T-S fuzzy systems. An outline is given on an important tool in control theory, Linear Matrix Inequalities and some standards LMI problems. Also, a brief recall on the state of the art of existing fuzzy control laws such as parallel distributed compensation (PDC), compensation and division for fuzzy models (CDF), state feedback control and fuzzy simultaneous stability (FSS). This chapter finishes with quadratic stabilization of T-S fuzzy models, especially, the continuous case which is considered in this thesis, starting by the example of the inverted pendulum given at the end of this chapter.

Chapter 4 is devoted to non-quadratic stability and stabilization of T-S fuzzy systems by Lyapunov method. Due to the limitation of the quadratic approach by the conservatism constraint, new non-quadratic stability conditions are proposed (Abdelmalek *et al.* , 2007). The control design is based on PDC concept. The new conditions are shown to be less conservative and allow stabilization of continuous T-S fuzzy systems including those that do not admit a quadratic stabilization.

Chapter 5 deals with the case of fuzzy control in presence of non measurable states. The fuzzy observer is designed separately from the fuzzy controller using the new non-quadratic stability conditions and applying a separation property to check the stability of the whole fuzzy system.

The thesis finishes with concluding remarks for this research and some prospects for the future.

The main contributions of this thesis are:

- A non-quadratic fuzzy stabilization and tracking approach to a two link robot manipulator (Abdelmalek & Goléa, 2006).
- A new fuzzy Lyapunov approach to non-quadratic stabilization of Takagi-Sugeno fuzzy models (Abdelmalek *et al.* , 2007).
- Fuzzy observer design for Takagi-Sugeno fuzzy models via Linear Matrix Inequalities (Abdelmalek & Goléa, 2007, a).
- Model-based fuzzy control of an inverted pendulum on a cart: fuzzy controller and fuzzy observer design via LMIs (Abdelmalek & Goléa, 2008).
- LMI-based design of fuzzy controller and fuzzy observer for continuous Takagi-Sugeno fuzzy systems: new non-quadratic stability approach (Abdelmalek & Goléa, 2009).

Chapter 2

Takagi-Sugeno Fuzzy Models

2.1 Introduction

Fuzzy sets theory and fuzzy logic provide the means for constructing fuzzy systems. Fuzzy sets were introduced by Professor L.A. Zadeh from berekley university in 1965 (Zadeh, 1965). Fuzzy logic provides a simple way to arrive at a definite conclusion based upon vague, ambiguous, imprecise, noisy, or missing input information. It is a powerful problem-solving methodology with a myriad of applications embedded in control and information processing since 70's years, such as Mamdani that concretizes the first time this method to realize a fuzzy control in an industrial application (Mamdani, 1974). Hence, unlike classical logic, that requires a deep understanding of a system, exact equations and precise numeric values, fuzzy logic incorporates an alternative way of thinking, which allows modeling complex systems using a higher level of abstraction originating from our knowledge and experience. The principal advantage of fuzzy logic systems is their aptitude to approximate any nonlinear function; they are universal approximators (Wang & Mendel, 1992). Takagi and Sugeno (Takagi & Sugeno, 1985) came up with an alternative rule format in order to make automated tuning possible and to reduce the number of fuzzy rules needed to construct the fuzzy model. Two particular structures of fuzzy systems will be detailed, followed by the advantages and the disadvantages of the one compared to the other.

In this chapter, a recall is given on modeling by two particular structures of fuzzy models that are Mamdani and T-S fuzzy models, followed by the advantages and the disadvantages of the one compared to the other. however, the construction procedure of a T-S fuzzy model is detailed. An outline is given on the concept of universal approximators, to show that a fuzzy model is able to approximate and then to represent any real function.

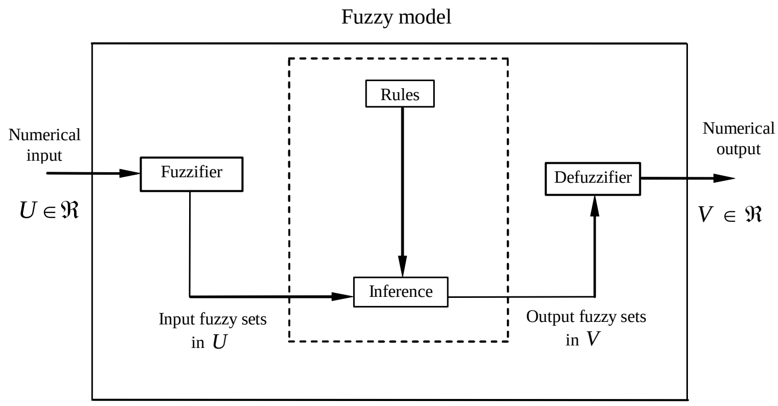

Figure 2.1: Fuzzy model structure

2.2 Fuzzy modeling

Fuzzy modeling is to construct a model through a description language based on fuzzy logic. It gives a qualitative description of systems functions and behaviors using a natural language. A fuzzy model consists, in general of four basic components: a fuzzy rule base, a fuzzy inference engine, a fuzzifier and a deffuzifier as shown in figure (2.1) (Wang, 1994),(Mendel, 1995),(Zhao, 1995).

- Fuzzy rule base is the knowledge base of the system to be modeled. It is a collection of IF-THEN rules, in general of the following form

  $$\text{Model Rule } i : \text{IF } z_1(t) \text{ is } M_{i1} \text{and ... and } z_p(t) \text{ is } M_{ip} \text{ THEN } y \text{ is } B^i \tag{2.1}$$

  where $z = [z_1, z_2, ..., z_p]^T \in \mathbb{R}^p$ is the input, y is the output and both are linguistic variables in the input product space $U_1 \times U_2 \times ... \times U_p$ and in the output space V. $M_{i1}, M_{i2}, ..., M_{ip}$ and B^i are fuzzy sets. $i \in [1, r]$, r is the number of rules in the fuzzy rule base.

- Fuzzy inference engine is a rule-based system that uses fuzzy logic, rather than Boolean logic, to reason about data. It simulates the human decision-making process by using fuzzy logic, and its task is to interpret and to construct an input-output mapping relationship with respect to all the rules.

- Fuzzifier performs the conversion from numerical values of input variables $z_1, z_2, ..., z_p$, obtained by sensors, into linguistic values represented as fuzzy sets $M_{i1}, M_{i2}, ..., M_{ip}$. Usually, a simple singleton fuzzifier is used.

- Defuzzifier performs the inverse conversion of that performed by the fuzzifier, i.e. conversion from the linguistic values in the form of fuzzy sets into non-fuzzy crisp values. Three kind of defuzzifier are employed in general: maximum defuzzifier, center average defuzzifier, and modified center average defuzzifier (Wang, 1994).

However, there exist two kinds of fuzzy models: Mamdani fuzzy models and T-S fuzzy models.

2.2.1 Mamdani fuzzy models

In this kind of fuzzy models, the fuzzy IF-THEN rules are of the following form:

$$\text{Rule } i: \text{IF } z_1(t) \text{ is } M_{i1} \text{and ... and } z_p(t) \text{ is } M_{ip} \text{ THEN } y \text{ is } B^i \tag{2.2}$$

The main advantages of this type of model are:

- the simplicity in representation of fuzzy rules,
- the flexibility in implementation ; due to the freedom to choose the operations included in fuzzy models.

The main disadvantage of this model is the great number of rules needed to represent a complex nonlinear system.

2.2.2 T-S fuzzy models

Takagi and Sugeno (Takagi & Sugeno, 1985) came up with the alternative rule format (2.3) in order to make automated tuning possible and to reduce the number of fuzzy rules. A T-S fuzzy model is described by fuzzy IF-THEN rules defined by the following

$$\begin{aligned} \text{Rule } i \quad &: \quad \text{IF } z_1(t) \text{ is } M_{i1} \text{ and ... and } z_p(t) \text{ is } M_{ip} \\ \text{THEN } y^i \quad &= \quad a_0^i + a_1^i z_1(t) + a_2^i z_2(t) + ... + a_p^i z_p(t) \end{aligned} \tag{2.3}$$

where $M_{i1}, ..., M_{ip}$ are fuzzy sets; $a_0^i, ..., a_p^i$ are the coefficients of the i-th linear consequent and is the output of the i-th fuzzy IF-THEN rule. The crisp output value of the T-S fuzzy model is a weighted average of the y^i is:

$$y = \frac{\sum_{i=1}^{r} w_i(t)\, y^i}{\sum_{i=1}^{r} w_i(t)} \tag{2.4}$$

where $0 \leq w_i(t) \leq 1$, $\sum_{i=1}^{r} w_i(t) > 0$ and $w_i(t) = \Pi_{j=1}^{p} M_{ij}(z_j(t))$, $M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij}. The equation (2.3) gives an affine T-S fuzzy model, for $a_0^i = 0$, we have a linear T-S model. T-S model represents a dynamical system whose IF-THEN rules represent local linear input-output relations of the nonlinear dynamical system. The main feature of a T-S fuzzy model is to express the local dynamics of each fuzzy rule by a linear sub-model, and then the overall fuzzy system is obtained by fuzzy "blending" of the linear sub-models (Tanaka & Wang, 2001, a).

The advantage of T-S fuzzy models is their powerful capability to represent a complex nonlinear relationship in spite of the smaller number of fuzzy IF-THEN rules, than that of Mamdani fuzzy models. These latter, with a centroid deffuzification, can be seen as a particular case of T-S fuzzy models. Moreover, in a T-S fuzzy model, the passage from one rule to another is done by a smooth transition between the rules, i.e., an interpolation. Thus (2.4) interpolates between different linear functions, that are the local models (figure (2.2)).

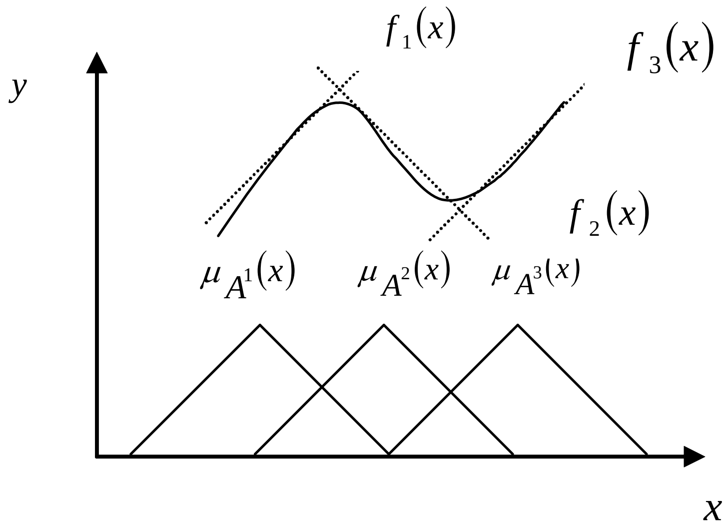

Figure 2.2: A function f defined by a T-S model

2.3 Construction of a T-S fuzzy model

In general, there exist two ways to construct a fuzzy model: by identification using input-output data (in other terms fuzzy modeling) or by derivation from a given nonlinear system equations.

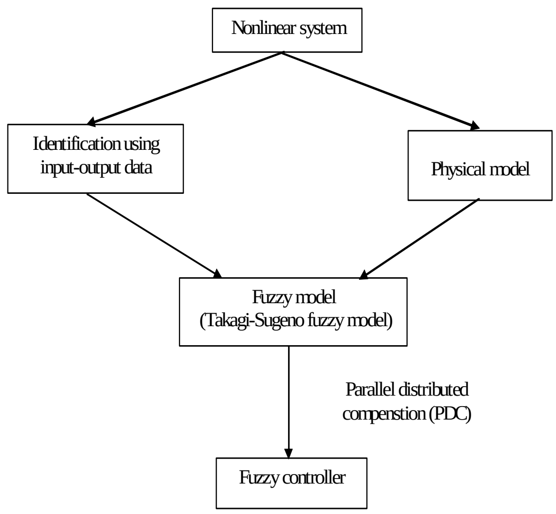

Figure 2.3: Model-based fuzzy control design

The identification approach is suitable for the modelization of plants that are difficult to represent using analytical models, whereas when the nonlinear dynamical models are available, the second approach is more appropriated (Tanaka & Wang, 2001, a). In both cases, we obtain a T-S fuzzy model whose *ith* rule form is:

$$\text{Rule } i : \text{IF } z_1(t) \text{ is } M_{i1} \text{ and ... and } z_p(t) \text{ is} M_{ip}$$

$$\text{THEN} \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \quad i = 1, 2, ..., r \tag{2.5}$$

The final outputs of the fuzzy model are inferred as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) (A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(z(t)) C_i x(t) \end{cases} \tag{2.6}$$

where $z(t) = [z_1(t), ..., z_p(t)]$ is the premise variable vector that may be functions of the state variables, measurable external disturbances and/or time. $A_i \in \Re^{n\times n}$, $B_i \in \Re^{n\times m}$, $C_i \in \Re^{q\times n}$, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input vector, $y(t) \in \Re^q$ is the output vector. r is the number of IF-THEN rules and M_{ij} is a fuzzy set. $h_i(z(t))$ is the normalized weight for each rule, that is

$$h_i(z(t)) \geq 0, \ \sum_{i=1}^{r} h_i(z(t)) = 1$$

and is given by:

$$h_i(z(t)) = \frac{w_i(t)}{\sum_{i=1}^{r} w_i(t)}$$

2.3.1 Identification approach

From input-output data, we obtain linear sub-models around the different operational points. The local linear sub-models, are fuzzy IF-THEN rules, whose consequent parts are linear models. This identification allows us to find an optimal model after estimating the parameters and validating, the final model (Sugeno & Kang, 1995),(Babuska, 1998). However, a state representation is used in the consequent part in order to extend the state feedback control principle to the nonlinear case.

2.3.2 Nonlinear dynamical model

When nonlinear dynamical models are easy to obtain, the linearization, the principle of sector nonlinearity or local approximation are more appropriated for constructing the fuzzy model (Tanaka & Wang, 2001, a).

Linearization

The basic idea is to linearize the nonlinear analytical model of the process about different operating points. Hence for the following nonlinear system

$$\dot{x}(t) = f(x(t), u(t)); \ f(\cdot) \in C^1 \tag{2.7}$$

The linearization of the system around an arbitrary operating point $(x_i, u_i) \in \mathbb{R}^n \times \mathbb{R}^p$, we have then:

$$\dot{x}(t) = A_i(x(t) - x_i) + B_i(u(t) - u_i) + f(x_i, u_i) \tag{2.8}$$

Taking $w_i = f(x_i, u_i) - A_i x_i - B_i u_i$, equation (2.8) can be rewritten:

$$\dot{x}(t) = A_i x(t) + B_i u(t) + w_i \tag{2.9}$$

where $A_i = \frac{\partial f(x,u)}{\partial x} |_{\substack{x=x_i \\ u=u_i}}$and $B_i = \frac{\partial f(x,u)}{\partial u} |_{\substack{x=x_i \\ u=u_i}}$

By considering that the local sub-models result from the linearization about r operational points, the T-S Fuzzy model is given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) - B_i u(t) + w_i) \tag{2.10}$$

Sector nonlinearity

The first apparition of sector non linearity in fuzzy model construction was in 1992 by Kawamoto et al. (Kawamoto *et al.* , 1992), it is based on considering a simple nonlinear system $\dot{x}(t) = f(x(t))$ where $f(0) = 0$. The objective is to find the global sector such that $\dot{x}(t) = f(x(t)) \in \begin{bmatrix} -a & a \end{bmatrix} x(t)$, as illustrated in figure 2.4. An exact fuzzy model construction is guaranteed with this method. However, it is sometimes difficult to find global sectors, then local sector nonlinearity is considered, where $x(t) \in \begin{bmatrix} -d & d \end{bmatrix}$. Figure 2.5 shows the local sector nonlinearity, where two lines become the local sectors under $-d < x(t) < d$. The nonlinear system is represented exactly by the fuzzy model in the "local" region $-d < x(t) < d$. But, it is often desirable to simplify the original nonlinear system as much as possible in order to reduce the number of rules. the following two examples illustrates this concept.

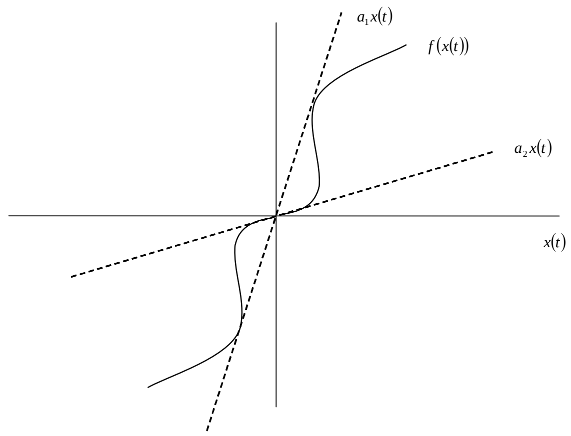

Figure 2.4: Global sector nonlinearity

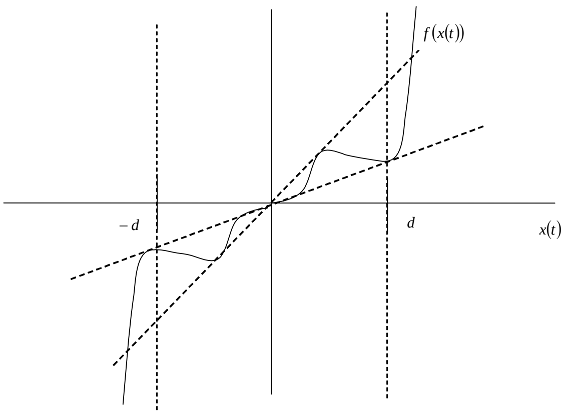

Figure 2.5: Local sector nonlinearity

Example 1

For the following nonlinear system (Tanaka & Wang, 2001, a):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1(t) + x_1(t)\, x_2^3(t) \\ -x_2(t) + (3 + x_2(t))\, x_1^3(t) \end{bmatrix} x(t) \qquad (2.11)$$

it is assumed for simplicity that $x_1(t) \in \begin{bmatrix} -1 & 1 \end{bmatrix}$ and $x_2(t) \in \begin{bmatrix} -1 & 1 \end{bmatrix}$, then (2.11) can be written as

$$\dot{x}(t) = \begin{bmatrix} -1 & x_1(t)\, x_2^2(t) \\ (3 + x_2(t))\, x_1^2(t) & -1 \end{bmatrix} x(t)$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ and $x_1(t)\, x_2^2(t)$ and $(3 + x_2(t))\, x_1^2(t)$ are nonlinear terms. By defining

$$z_1(t) \equiv x_1(t)\, x_2^2(t) \text{ and } z_2(t) \equiv (3 + x_2(t))\, x_1^2(t)$$

we have:

$$\dot{x}(t) = \begin{bmatrix} -1 & z_1(t) \\ z_2(t) & -1 \end{bmatrix} x(t).$$

Then, the minimum and maximum values are calculated under $x_1(t) \in \begin{bmatrix} -1 & 1 \end{bmatrix}$ and $x_2(t) \in \begin{bmatrix} -1 & 1 \end{bmatrix}$, their values are:

$$\max_{x_1(t), x_2(t)} z_1(t) = 1, \quad \min_{x_1(t), x_2(t)} z_1(t) = -1$$

$$\max_{x_1(t), x_2(t)} z_2(t) = 4, \quad \min_{x_1(t), x_2(t)} z_2(t) = 0$$

From these values, $z_1(t)$ and $z_2(t)$ can be represented by:

$$\begin{aligned} z_1(t) &\equiv x_1(t)\,x_2^2(t) = M_1(z_1(t))\cdot 1 + M_2(z_1(t))\cdot(-1) \\ z_2(t) &\equiv (3+x_2(t))\,x_1^2(t) = N_1(z_2(t))\cdot 4 + N_2(z_2(t))\cdot 0 \end{aligned}$$

where

$$M_1(z_1(t)) + M_2(z_1(t)) = 1$$

$$N_1(z_2(t)) + N_2(z_2(t)) = 1$$

Therefore the membership functions can be calculated by:

$$M_1(z_1(t)) = \frac{z_1(t)+1}{2}, \quad M_2(z_1(t)) = \frac{1-z_1(t)}{2}$$

$$N_1(z_2(t)) = \frac{z_2(t)}{4}, \; N_2(z_2(t)) = \frac{4-z_2(t)}{4}$$

These membership functions are named respectively: "Positive", "Negative", "Big" and "Small". Hence, equation (2.11) can be represented by the following fuzzy model:

$$\text{Rule 1}: \text{IF } z_1(t) \text{ is } "Positive" \text{ and } z_2(t) \text{ is } "Big" \text{ THEN } \dot{x}(t) = A_1 x(t)$$

$$\text{Rule 2}: \text{IF } z_1(t) \text{ is } "Positive" \text{ and } z_2(t) \text{ is } "Small" \text{ THEN } \dot{x}(t) = A_2 x(t)$$

$$\text{Rule 3}: \text{IF } z_1(t) \text{ is } "Negative" \text{ and } z_2(t) \text{ is } "Big" \text{ THEN } \dot{x}(t) = A_3 x(t)$$

$$\text{Rule 4}: \text{IF } z_1(t) \text{ is } "Negative" \text{ and } z_2(t) \text{ is } "Small" \text{ THEN } \dot{x}(t) = A_4 x(t)$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \\ A_3 &= \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}, A_4 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

The deffuzification yields:

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(z(t))\,A_i x(t)$$

where

$$\begin{aligned} h_1(z(t)) &= M_1(z_1(t)) \times N_1(z_2(t)) \\ h_2(z(t)) &= M_1(z_1(t)) \times N_2(z_2(t)) \\ h_3(z(t)) &= M_2(z_1(t)) \times N_1(z_2(t)) \\ h_4(z(t)) &= M_2(z_1(t)) \times N_2(z_2(t)) \end{aligned}$$

Finally, this model represents the nonlinear system in the region $\begin{bmatrix} -1 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \end{bmatrix}$ of the $x_1 - x_2$ space.

Example 2

For the inverted pendulum defined by the following equations of motion (Tanaka & Wang, 2001, a):

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= \frac{g\sin(x_1(t)) - amlx_2^2(t)\sin(2(x_1(t)))/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))}
\end{aligned} \tag{2.12}$$

where $x_1(t)$ denotes the angle (in radians) of the pendulum from the vertical and $x_2(t)$ is the angular velocity, $g = 9.8\ m/s^2$ is the gravity constant, m is the mass of the pendulum. M is the mass of the cart, $2l$ is the length of the pendulum, u is the force applied to the cart (in newtons) and $a = 1/(m+M)$. Equation (2.12) is rewritten as

$$\begin{aligned}
\dot{x}_2(t) &= \frac{gx_1(t) - au(t)}{4l/3 - aml} \\
&\times \left(g\sin(x_1(t)) - \frac{amlx_2(t)\sin(2x_1(t))}{2}x_2(t) - a\cos(x_1(t))u(t)\right)
\end{aligned} \tag{2.13}$$

Define

$$\begin{aligned}
z_1(t) &\equiv \frac{1}{4l/3 - aml\cos^2(x_1(t))} \\
z_2(t) &\equiv \sin(x_1(t)) \\
z_3(t) &\equiv x_2(t)\sin(2x_1(t)) \\
z_4(t) &\equiv \cos(x_1(t))
\end{aligned}$$

where $x_1(t) \in (-\pi/2, \pi/2)$ and $x_2(t) \in \begin{bmatrix} -\alpha & \alpha \end{bmatrix}$. To maintain controllability of the fuzzy model, we assume that $x_1(t) \in [-88°, 88°]$. Equation (2.13) is rewritten as

$$\dot{x}_2(t) = z_1(t)\left\{gz_2(t) - \frac{aml}{2}z_3(t)x_2(t) - az_4(t)u(t)\right\}$$

we replace $z_1(t) - z_4(t)$ with T-S fuzzy model representation. Since

$$\begin{aligned}
\max z_1(t) &= \frac{1}{4l/3 - aml\beta^2} \equiv q_1,\ \beta = \cos(88°), \\
\min z_1(t) &= \frac{1}{4l/3 - aml} \equiv q_2,
\end{aligned}$$

$z_1(t)$ can be rewritten as

$$z_1(t) = \sum_{i=1}^{2} E_i(z_1(t))q_i \tag{2.14}$$

where

$$E_1(z_1(t)) = \frac{z_1(t) - q_2}{q_1 - q_2}, \; E_2(z_1(t)) = \frac{q_1 - z_1(t)}{q_1 - q_2}$$

These membership functions are obtained from the property of $E_1(z_1(t)) + E_2(z_1(t)) = 1$. Figure 2.6 shows the local sector of $z_2(t) = \sin(x_1(t))$ for $x_1(t) \in (-\pi/2, \pi/2)$. The sector $[b_1, b_2]$ consists of two lines $b_1 x_1$ and $b_2 x_1$, where $b_1 = 1$ and $b_2 = 2/\pi$ are the slopes

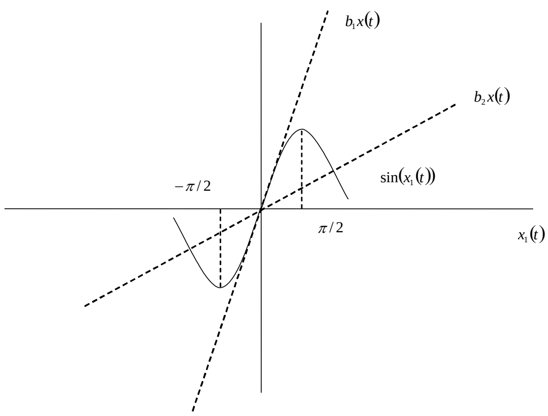

Figure 2.6: $\sin(x_1(t))$ and its sector

Therefore $\sin(x_1(t))$ is represented as follows:

$$z_2(t) = \sin(x_1(t)) = \left(\sum_{i=1}^{2} M_i(z_2(t))\, b_i \right) x_1(t) \tag{2.15}$$

Then, from $[M_1(z_2(t)) + M_2(z_2(t)) = 1]$, the membership functions are

$$M_1(z_2(t)) = \begin{cases} \frac{z_2(t) - (2/\pi)\sin^{-1}(z_2(t))}{(1-2/\pi)\sin^{-1}(z_2(t))}, & z_2(t) \neq 0 \\ 1, & otherwise \end{cases}$$

$$M_2(z_2(t)) = \begin{cases} \frac{(2/\pi)\sin^{-1}(z_2(t)) - z_2(t)}{(1-2/\pi)\sin^{-1}(z_2(t))}, & z_2(t) \neq 0 \\ 0, & otherwise \end{cases}$$

We consider next, $z_3(t) = x_2(t)\sin(2x_1(t))$. Since

$$\max_{x_1(t), x_2(t)} z_3(t) = \alpha \equiv c_1 \text{ and } \min_{x_1(t), x_2(t)} z_3(t) = -\alpha \equiv c_2$$

In the same way as for the $z_1(t)$ case:

$$z_3(t) = x_2(t)\sin(2x_1(t)) = \sum_{i=1}^{2} N_i(z_3(t))\, c_i \tag{2.16}$$

where

$$N_1(z_3(t)) = \frac{z_3(t) - c_2}{c_1 - c_2},\ N_2(z_3(t)) = \frac{c_1 - z_3(t)}{c_1 - c_2}$$

The same procedure is applied for $z_4(t)$. Since

$$\max_{x_1(t)} z_4(t) = 1 \equiv d_1 \text{ and } \min_{x_1(t)} z_4(t) = \beta \equiv d_2$$

we obtain

$$z_4(t) = \cos(x_1(t)) = \sum_{i=1}^{2} S_i(z_4(t))\, d_i \tag{2.17}$$

where

$$S_1(z_4(t)) = \frac{z_4(t) - d_2}{d_1 - d_2},\ S_2(z_4(t)) = \frac{d_1 - z_4(t)}{d_1 - d_2}$$

From (2.14)-(2.17), the following T-S fuzzy model is constructed for the inverted pendulum:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} E_i(z_1(t))\, M_j(z_2(t))\, N_k(z_3(t))\, S_l(z_4(t)) \\
&\quad \times \left(\begin{bmatrix} 0 & 1 \\ g q_i b_j & -\frac{aml}{2} q_i c_k \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -a q_i d_l \end{bmatrix} \right) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} E_i(z_1(t))\, M_j(z_2(t))\, N_k(z_3(t))\, S_l(z_4(t)) \\
&\quad \times \{A_{ijkl} x(t) + B_{ijkl} u(t)\}
\end{aligned} \tag{2.18}$$

The summations in (2.18) can be aggregated as one summation:

$$\dot{x}(t) = \sum_{\rho=1}^{16} h_\rho(z(t)) \left\{A_\rho^* x(t) + B_\rho^* u(t)\right\} \tag{2.19}$$

where

$$\rho = l + 2(k-1) + 4(j-1) + 8(i-1),$$

$$h_\rho(z(t)) = E_i(z_1(t))\, M_j(z_2(t))\, N_k(z_3(t))\, S_l(z_4(t)),$$

$$A_\rho^* = A_{ijkl},\ B_\rho^* = B_{ijkl}$$

Equation (2.19) means that the fuzzy model has the following 16 rules:

$$\text{Rule 1:} \begin{cases} \text{IF } z_1(t) \text{ is "}Positive\text{" and } z_2(t) \text{ is "}Zero\text{"} \\ \text{and } z_3(t) \text{ is "}Positive\text{" and } z_4(t) \text{ is "}Big\text{"} \\ \text{THEN} \dot{x}(t) = A_1^* x(t) + B_1^* u(t) \end{cases}$$

$$\text{Rule 2:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Positive}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_2^* x(t) + B_2^* u(t)\end{cases}$$

$$\text{Rule 3:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Big}\text{"}\\ \text{THEN}\dot{x}(t) = A_3^* x(t) + B_3^* u(t)\end{cases}$$

$$\text{Rule 4:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_4^* x(t) + B_4^* u(t)\end{cases}$$

$$\text{Rule 5:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Not\ Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Positive}\text{" and } z_4(t) \text{ is "}\mathit{Big}\text{"}\\ \text{THEN}\dot{x}(t) = A_5^* x(t) + B_5^* u(t)\end{cases}$$

$$\text{Rule 6:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Not\ Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Positive}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_6^* x(t) + B_6^* u(t)\end{cases}$$

$$\text{Rule 7:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Not\ Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Big}\text{"}\\ \text{THEN}\dot{x}(t) = A_7^* x(t) + B_7^* u(t)\end{cases}$$

$$\text{Rule 8:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Positive}\text{" and } z_2(t) \text{ is "}\mathit{Not\ Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_8^* x(t) + B_8^* u(t)\end{cases}$$

$$\text{Rule 9:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Negative}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Positive}\text{" and } z_4(t) \text{ is "}\mathit{Big}\text{"}\\ \text{THEN}\dot{x}(t) = A_9^* x(t) + B_9^* u(t)\end{cases}$$

$$\text{Rule 10:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Negative}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Positive}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_{10}^* x(t) + B_{10}^* u(t)\end{cases}$$

$$\text{Rule 11:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Negative}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Big}\text{"}\\ \text{THEN}\dot{x}(t) = A_{11}^* x(t) + B_{11}^* u(t)\end{cases}$$

$$\text{Rule 12:}\begin{cases}\text{IF } z_1(t) \text{ is "}\mathit{Negative}\text{" and } z_2(t) \text{ is "}\mathit{Zero}\text{"}\\ \text{and } z_3(t) \text{ is "}\mathit{Negative}\text{" and } z_4(t) \text{ is "}\mathit{Small}\text{"}\\ \text{THEN}\dot{x}(t) = A_{12}^* x(t) + B_{12}^* u(t)\end{cases}$$

$$\text{Rule 13:}\begin{cases}\text{IF } z_1(t) \text{ is "}Negative\text{" and } z_2(t) \text{ is "}Not\ Zero\text{"}\\ \text{and } z_3(t) \text{ is "}Positive\text{" and } z_4(t) \text{ is "}Big\text{"}\\ \text{THEN}\dot{x}(t) = A_{13}^{*}x(t) + B_{13}^{*}u(t)\end{cases}$$

$$\text{Rule 14:}\begin{cases}\text{IF } z_1(t) \text{ is "}Negative\text{" and } z_2(t) \text{ is "}Not\ Zero\text{"}\\ \text{and } z_3(t) \text{ is "}Positive\text{" and } z_4(t) \text{ is "}Small\text{"}\\ \text{THEN}\dot{x}(t) = A_{14}^{*}x(t) + B_{14}^{*}u(t)\end{cases}$$

$$\text{Rule 15:}\begin{cases}\text{IF } z_1(t) \text{ is "}Negative\text{" and } z_2(t) \text{ is "}Not\ Zero\text{"}\\ \text{and } z_3(t) \text{ is "}Negative\text{" and } z_4(t) \text{ is "}Big\text{"}\\ \text{THEN}\dot{x}(t) = A_{15}^{*}x(t) + B_{15}^{*}u(t)\end{cases}$$

$$\text{Rule 16:}\begin{cases}\text{IF } z_1(t) \text{ is "}Negative\text{" and } z_2(t) \text{ is "}Not\ Zero\text{"}\\ \text{and } z_3(t) \text{ is "}Negative\text{" and } z_4(t) \text{ is "}Small\text{"}\\ \text{THEN}\dot{x}(t) = A_{16}^{*}x(t) + B_{16}^{*}u(t)\end{cases}$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ are premise variables and

$$A_1^* = A_{1111} = \begin{bmatrix} 0 & 1 \\ gq_1b_1 & -\frac{aml}{2}q_1c_1 \end{bmatrix},\ B_1^* = B_{1111} = \begin{bmatrix} 0 \\ -aq_1d_1 \end{bmatrix},$$

$$A_2^* = A_{1112} = \begin{bmatrix} 0 & 1 \\ gq_1b_1 & -\frac{aml}{2}q_1c_1 \end{bmatrix},\ B_2^* = B_{1112} = \begin{bmatrix} 0 \\ -aq_1d_2 \end{bmatrix},$$

$$A_3^* = A_{1121} = \begin{bmatrix} 0 & 1 \\ gq_1b_1 & -\frac{aml}{2}q_1c_2 \end{bmatrix},\ B_3^* = B_{1121} = \begin{bmatrix} 0 \\ -aq_1d_1 \end{bmatrix},$$

$$A_4^* = A_{1122} = \begin{bmatrix} 0 & 1 \\ gq_1b_1 & -\frac{aml}{2}q_1c_2 \end{bmatrix},\ B_4^* = B_{1122} = \begin{bmatrix} 0 \\ -aq_1d_2 \end{bmatrix},$$

$$A_5^* = A_{1211} = \begin{bmatrix} 0 & 1 \\ gq_1b_2 & -\frac{aml}{2}q_1c_1 \end{bmatrix},\ B_5^* = B_{1211} = \begin{bmatrix} 0 \\ -aq_1d_1 \end{bmatrix},$$

$$A_6^* = A_{1212} = \begin{bmatrix} 0 & 1 \\ gq_1b_2 & -\frac{aml}{2}q_1c_1 \end{bmatrix},\ B_6^* = B_{1212} = \begin{bmatrix} 0 \\ -aq_1d_2 \end{bmatrix},$$

$$A_7^* = A_{1221} = \begin{bmatrix} 0 & 1 \\ gq_1b_2 & -\frac{aml}{2}q_1c_2 \end{bmatrix},\ B_7^* = B_{1221} = \begin{bmatrix} 0 \\ -aq_1d_1 \end{bmatrix},$$

$$A_8^* = A_{1222} = \begin{bmatrix} 0 & 1 \\ gq_1b_2 & -\frac{aml}{2}q_1c_2 \end{bmatrix},\ B_8^* = B_{1222} = \begin{bmatrix} 0 \\ -aq_1d_2 \end{bmatrix},$$

$$A_9^* = A_{2111} = \begin{bmatrix} 0 & 1 \\ gq_2b_1 & -\frac{aml}{2}q_2c_1 \end{bmatrix},\ B_9^* = B_{2111} = \begin{bmatrix} 0 \\ -aq_2d_1 \end{bmatrix},$$

$$A_{10}^* = A_{2112} = \begin{bmatrix} 0 & 1 \\ gq_2b_1 & -\frac{aml}{2}q_2c_1 \end{bmatrix},\ B_{10}^* = B_{2112} = \begin{bmatrix} 0 \\ -aq_2d_2 \end{bmatrix},$$

$$A_{11}^* = A_{2121} = \begin{bmatrix} 0 & 1 \\ gq_2b_1 & -\frac{aml}{2}q_2c_2 \end{bmatrix},\ B_{11}^* = B_{2121} = \begin{bmatrix} 0 \\ -aq_2d_1 \end{bmatrix},$$

$$A_{12}^* = A_{2122} = \begin{bmatrix} 0 & 1 \\ gq_2b_1 & -\frac{aml}{2}q_2c_2 \end{bmatrix},\ B_{12}^* = B_{2122} = \begin{bmatrix} 0 \\ -aq_2d_2 \end{bmatrix},$$

$$A_{13}^* = A_{2211} = \begin{bmatrix} 0 & 1 \\ gq_2b_2 & -\frac{aml}{2}q_2c_1 \end{bmatrix},\ B_{13}^* = B_{2211} = \begin{bmatrix} 0 \\ -aq_2d_1 \end{bmatrix},$$

$$A_{14}^* = A_{2212} = \begin{bmatrix} 0 & 1 \\ gq_2b_2 & -\frac{aml}{2}q_2c_1 \end{bmatrix},\ B_{14}^* = B_{2212} = \begin{bmatrix} 0 \\ -aq_2d_2 \end{bmatrix},$$

$$A_{15}^* = A_{2221} = \begin{bmatrix} 0 & 1 \\ gq_2b_2 & -\frac{aml}{2}q_2c_2 \end{bmatrix},\ B_{15}^* = B_{2221} = \begin{bmatrix} 0 \\ -aq_2d_1 \end{bmatrix},$$

$$A_{16}^* = A_{2222} = \begin{bmatrix} 0 & 1 \\ gq_2b_2 & -\frac{aml}{2}q_2c_2 \end{bmatrix},\ B_{16}^* = B_{2222} = \begin{bmatrix} 0 \\ -aq_2d_2 \end{bmatrix}$$

Figures 2.7- 2.10 show the membership functions, that is

$$E_1(z_1(t)) = \frac{z_1(t) - q_2}{q_1 - q_2},\ E_2(z_1(t)) = \frac{q_1 - z_1(t)}{q_1 - q_2}$$

$$M_1(z_2(t)) = \frac{\sin(x_1(t)) - (2/\pi)\, z_2(t)}{(1 - 2/\pi)\, z_2(t)},\ M_2(z_2(t)) = \frac{x_1(t) - z_2(t)}{(1 - 2/\pi)\, z_2(t)}$$

$$N_1(z_3(t)) = \frac{z_3(t) - c_2}{c_1 - c_2},\ N_2(z_3(t)) = \frac{c_1 - z_3(t)}{c_1 - c_2}$$

$$S_1(z_4(t)) = \frac{z_4(t) - d_2}{d_1 - d_2},\ S_2(z_4(t)) = \frac{d_1 - z_4(t)}{d_1 - d_2}$$

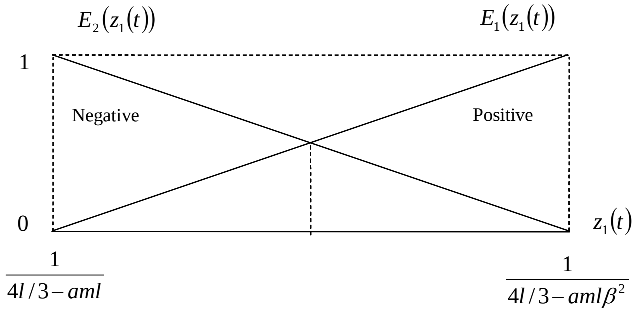

Figure 2.7: Membership functions $E_1(z_1(t))$ and $E_2(z_1(t))$

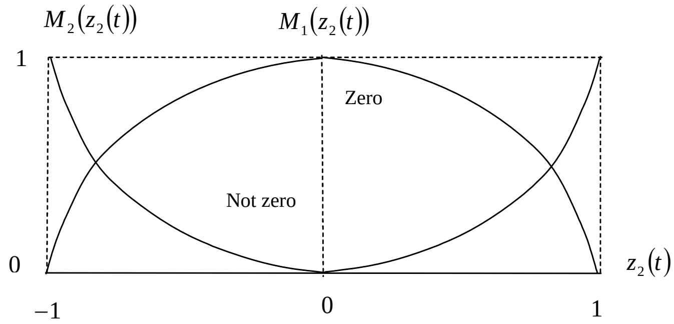

Figure 2.8: Membership functions $M_1(z_2(t))$ and $M_2(z_2(t))$

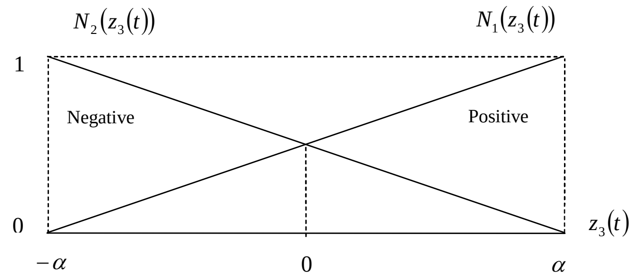

Figure 2.9: Membership functions $N_1(z_3(t))$ and $N_2(z_3(t))$

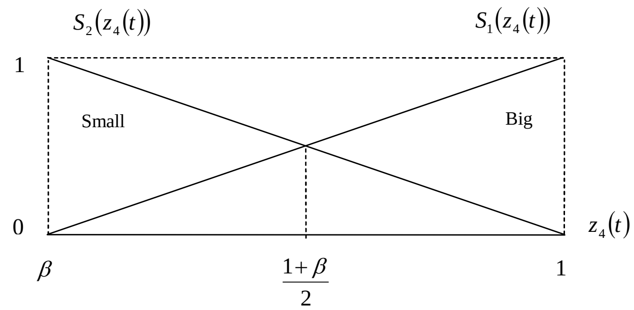

Figure 2.10: Membership functions $S_1(z_4(t))$ and $S_4(z_3(t))$

Local approximation in fuzzy partition spaces

The principle of this method is to approximate nonlinear terms by adequate chosen linear terms, that leads to a less number of rules. For example, Tanaka (Tanaka & Wang, 2001, a) proposed in his book a fuzzy modelization of an inverted pendulum with 16 rules using the sector nonlinearity method, whereas, using the local approximation, the inverted pendulum is represented by a two rules fuzzy model.

Example 3

For the inverted pendulum defined by equations of motion (2.12), the simplification leads to two cases:

when $x_1(t)$ is near zero we have:

$$\dot{x}_1(t) = x_2(t), \tag{2.20}$$

$$\dot{x}_2(t) = \frac{gx_1(t) - au(t)}{4l/3 - aml} \tag{2.21}$$

whereas when $x_1(t)$ is near $\pm\pi/2$ we have:

$$\dot{x}_1(t) = x_2(t), \tag{2.22}$$

$$\dot{x}_2(t) = \frac{2gx_1(t)/\pi - a\beta u(t)}{4l/3 - aml\beta^2} \tag{2.23}$$

where $\beta = \cos(88°)$ .The equations (2.20)-(2.23) are linear systems that produces the following fuzzy modelization of the inverted pendulum:

Model Rule 1 : IF $x_1(t)$ is about 0 THEN $\dot{x}(t) = A_1x(t) + B_1u(t)$

Model Rule 2 : IF $x_1(t)$ is about $\pm\pi/2$ $(|x_1| < \pi/2)$ THEN $\dot{x}(t) = A_2x(t) + B_2u(t)$

where the membership functions are of triangular types

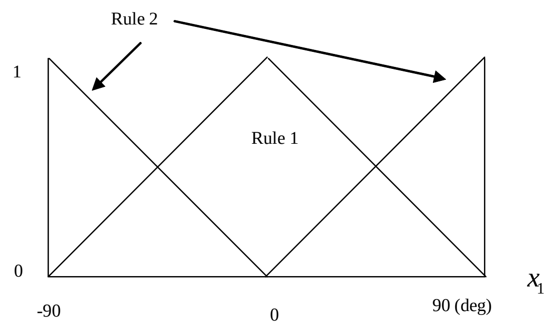

Figure 2.11: Membership functions of two rules model

and

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{4l/3-aml} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi\left(4l/3-aml\beta^2\right)} & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{bmatrix}, \beta = \cos\left(88^\circ\right)$$

2.4 Fuzzy systems as universal approximators

A fuzzy system can be regarded as a (multidimensional) input-output mapping $y = f(x)$. Several authors have proved that given enough rules, the fuzzy system can approximate any real continuous function to any given accuracy. The following theorem shows that the fuzzy logic system

$$f\left(x\right) = \frac{\sum_{l=1}^{r} \bar{y}_l \left[\Pi_{i=1}^{n} a_i^l \exp\left(-\left(\frac{x_i-\bar{x}_i^l}{\sigma_i^l}\right)^2\right)\right]}{\sum_{l=1}^{r} \left[\Pi_{i=1}^{n} a_i^l \exp\left(-\left(\frac{x_i-\bar{x}_i^l}{\sigma_i^l}\right)^2\right)\right]} \tag{2.24}$$

with center average defuzzifier, product inference rule, singleton fuzzifier and gaussian membership function, is able to uniformly approximate any nonlinear function over U to any degree of accuracy.

Theorem 1 *Universal approximation theorem (Wang & Mendel, 1992): For any given real continuous function g on a compact set $U \subset \mathbb{R}^n$ and arbitrary $\epsilon > 0$, there exists a fuzzy logic system f in the form of (2.24) such that*

$$\sup_{x\in U} \left|f\left(x\right) - g\left(x\right)\right| < \epsilon \tag{2.25}$$

This theorem provides a justification for applying the fuzzy logic systems to almost any nonlinear modeling problems.

2.5 conclusion

This chapter has recalled in first the two types of fuzzy models representations for complex systems, that are Mamdani fuzzy models and T-S fuzzy models. An attention is given to T-S fuzzy models due to their interesting characteristics. Different existing methods for constructing a T-S fuzzy model are detailed and illustrated by different examples. This chapter finishes by a theorem on the concept of universal approximators of T-S fuzzy models.

Chapter 3

Quadratic stability and Stabilization of T-S Fuzzy Systems

3.1 Introduction

This chapter deals with the fuzzy stability and stabilization of T-S fuzzy systems. During the last decade, several researchers in the control community have come up with different techniques for designing control systems. Fuzzy control is probably one of the most popular (Takagi & Sugeno, 1985),(Tanaka & Sugeno, 1992),(Wang *et al.* , 1996),(Feng, 2002), since it can provide an effective solution to the control of plants that are complex, uncertain or ill-defined, by combining the flexibility of fuzzy logic theory and the rigorous mathematical analysis tools in linear system theory into a unified framework. Tanaka and Sugeno (Tanaka & Sugeno, 1992) showed that finding common symmetric positive definite matrix P for N sub-systems could show the stability of continuous T-S fuzzy system. Generally, most of the stability criteria for this fuzzy system derived by Lyapunov approach needs a common P to satisfy a set of Lyapunov inequalities (Tanaka *et al.* , 1996),(Wang *et al.* , 1996). These inequalities can be reduced to LMI problems, that can be solved efficiently in practice by convex programming techniques for LMIs. The fuzzy controller is based on the PDC design with the principle of deriving each control rule from the corresponding rule of T-S fuzzy model so as to compensate it.

This chapter considers the stability and stabilization analysis of continuous T-S fuzzy models using Lyapunov approach, PDC and LMIs. LMIs are detailed in this chapter with some standard LMI problems used in control applications and an outline on the different existing fuzzy control laws is given. Finally, the inverted pendulum example illustrates these different concepts.

3.2 Systems stability

The stability of systems in closed loop is one of the most significant problems in control theory. The T-S systems stability analysis was the objective of several works (Tanaka & Sugeno, 1992),(Wang *et al.* , 1996),(Jadbabaie, 1997, b),(Tanaka *et al.* , 1998),(Kim & Lee, 2000),(Manamanni *et al.* , 2007) by quadratic Lyapunov functions (Johansson *et al.* , 1999, a),(Rantzer & Johansson, 2000),(Chadli *et al.* , 2003),(Ohtake *et al.* , 2003),(Feng, 2004) by piecewise quadratic Lyapunov functions and (Blanco *et al.* , 2001),(Tanaka *et al.* , 2001, c),(Chadli *et al.* , 2002),(Tanaka *et al.* , 2003),(Teixeira *et al.* , 2003),(Guerra & Vermeiren, 2004),(Bernal & Hušek, 2005),(Zhou *et al.* , 2007) by non-quadratic Lyapunov functions. In the majority, the goal was the obtention of a global asymptotic stability by applying Lyapunov's direct method based on Lyapunov functions, which measure the system's energy. Stability in the Lyapunov sense is a mathematical translation of an elementary observation: if the total energy of a system dissipates in a continuous manner (i.e. decreases with time), then this system (that it is linear or not, stationary or not) tends to an equilibrium state (it is stable). The direct method thus seeks to generate a scalar function of energy type which admits a negative temporal derivative. There exist some definitions related to Lyapunov stability, among them the following.

Definition 2 *The equilibrium point x_e is stable if*

$$\forall t \geq 0, \forall \epsilon > 0, \exists \alpha > 0 \textit{ such that } \|x(0) - x_e\| < \alpha \Rightarrow \|x(t) - x_e\| < \epsilon$$

in the contrary case x_e is unstable.

In other terms, a system is stable in Lyapunov sense if and only if a weak disturbance of the initial conditions involves a weak disturbance of the system trajectory. Another important definition in system control theory is the global asymptotic stability.

Definition 3 *The equilibrium point $x_e = 0$ is locally asymptotically stable if it is stable and there exist $r > 0$ such that:*

$$\textit{if } \|x(0)\| < r \textit{ then } lim_{t\to\infty} \|x(t)\| \to 0$$

Definition 4 *if a system is asymptotically stable for any initial condition in $\mathbb{R}^n$, then $x_e = 0$ is asymptotically stable in the large or globally asymptotically stable (GAS).*

3.3 Lyapunov functions in the control literature

In general, there is no a systematic method to find candidate Lyapunov functions. The degree of conservatism of the obtained stability conditions depends on the Lyapunov function form and the system structure. Different Lyapunov functions forms are used by different authors in the literature (Tanaka & Sano, 1994),(Wang *et al.* , 1996),(Tanaka *et al.* , 1998),(Wong *et al.* , 1998),(Jadbabaie, 1999),(Johansson *et al.* , 1999, a),(Chadli *et al.* , 2000),(Blanco *et al.* , 2001),(Chadli *et al.* , 2001),(Ohtake *et al.* , 2003),(Tanaka *et al.* , 2003),(Bernal & Hušek, 2005), depending of the system nature and complexity.

3.3.1 Quadratic Lyapunov function

This one is the classical form, it is given by:

$$V\left(x\left(t\right)\right)=x^{T}\left(t\right)Px\left(t\right),\;\;P>0,\;\;P^{T}=P \tag{3.1}$$

used initially to stability study of linear systems and then for MIMO nonlinear systems, in particular, T-S fuzzy systems (Tanaka & Sugeno, 1992),(Zhao, 1995),(Wang *et al.* , 1996),(Tanaka *et al.* , 1998). The principle of the method is to search a positive definite matrix P, by the way of convex formulation of the problem. The drawback of this quadratic approach is the conservative stability conditions, but it remains from a practical point of view easy to implement.

3.3.2 Non-quadratic Lyapunov function

This function is of the form:

$$V\left(x\left(t\right)\right)=\sum_{i=1}^{r}h_{i}\left(z\left(t\right)\right)x^{T}\left(t\right)P_{i}x\left(t\right) \tag{3.2}$$

where P_i is a positive definite matrix and $h_i\left(z\left(t\right)\right)\geq 0$, $\sum_{i=1}^{r}h_i\left(z\left(t\right)\right)=1$. It is a more general function since it includes the quadratic case when $P_i = P$, $i = 1, .., r$. However, an interesting advantage is that, the non-quadratic form of Lyapunov function takes into account the speed variation of the decision variables, what allows the conservatism reduction and more relaxed stability conditions. Indeed, it has been studied by many authors, (Jadbabaie, 1999),(Chadli *et al.* , 2000),(Morère, 2001),(Tanaka *et al.* , 2001, d), who concluded on the need to have an upper bounds on the speed variations of decision variables and then on the first time derivative of premise membership functions to reduce conservatism. On another side, this type of functions reduces the global stability of the nonlinear system to the analysis of the local stability of each local linear model (sub-model) separately. However, this Lyapunov function was also used in the discrete case by several authors such as (Daafouz & Bernussou, 2001),(Morère, 2001).

3.3.3 Piecewise quadratic Lyapunov function

T-S fuzzy systems and affine T-S fuzzy systems can be considered as piecewise linear systems. Hence, many authors such as (Johansson, 1999),(Johansson *et al.* , 1999, a),(Feng, 2002),(Ohtake *et al.* , 2003) proposed piecewise quadratic Lyapunov functions to reduce conservatism and whose search is based on a convex optimization problem, they are given by:

$$V(x) = \begin{cases} x^T P_i x, , \quad x \in X_i, \quad i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in X_i, \quad i \in I_1 \end{cases} \tag{3.3}$$

where the operating region X_i is a partition in the state space and corresponds to a dynamic local model i. Thus, the principle of the approach is to divide the space into two regions: an operating region and an interpolation region. This Lyapunov function combines the power of quadratic Lyapunov functions near an equilibrium point with the flexibility of piecewise linear functions in the large. It also allows conservatism reduction because of the space partionning induced by a locally bounded membership function. This leads to search for a common P_i to all active local linear models in each region. However, the conservatism reappears for this approach when the number of activated local models becomes equal to the total number of local models.

3.4 Quadratic stability of Takagi-Sugeno fuzzy systems

In this section quadratic Lyapunov functions are considered whose one of the existing definitions is:

Definition 5 *The system $\dot{x}(t) = f(x(t), u(t))$ is said to be quadratically stable if there exists a quadratic function $V(x(t)) = x^T(t) Px(t)$, $V(0) = 0$, satisfying the following conditions:*

$$V(x(t)) > 0 \ \forall x(t) \neq 0 \Longleftrightarrow P > 0, \tag{3.4}$$

$$\dot{V}(x(t)) < 0 \ \forall x(t) \neq 0. \tag{3.5}$$

If V exists, it is called a Lyapunov function.

Thus, to find a Lyapunov function amounts finding a positive definite matrix P, we speak about quadratic stability. The following stability theorem that is based on quadratic Lyapunov functions give sufficient conditions to assure stability of the open loop T-S fuzzy system given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t), \tag{3.6}$$

Theorem 6 *(Tanaka & Sugeno, 1992)The equilibrium of a fuzzy system (3.6) is globally asymptotically stable if there exists a common positive definite matrix P such that*

$$A_i^T P + P A_i < 0, \quad i = 1, ..., r$$

that is, a common P has to exist for all sub-models (Tanaka & Sugeno, 1992),(Tanaka & Sano, 1995),(Tanaka & Wang, 2001, a). The proof of theorem 6 is given in appendix A.

This theorem presents sufficient conditions for the quadratic stability. However, they are conservative since the $h_i(z(t))$ are not taken into account. The common P problem can be solved efficiently via convex optimization techniques and LMIs for Linear Matrix Inequalities; we call this an LMI feasibility problem. Therefore, recasting a control problem (such as stabilization via PDC controller) as an LMI problem is equivalent to finding a "solution" to the original problem. The existence of P depends on two conditions: the first one is related to the stability of all sub-models, where each matrix A_i must be Hurwitz. The second condition relates to the existence of a common Lyapunov function for the the r sub-models. It requires that $\sum_{i=1}^{r} A_i$ must also be Hurwitz. However if r, that is the number of IF-THEN rules, is large, it might be difficult to find a common P.

3.5 Fuzzy control laws

In the literature, different control laws were proposed to stabilize fuzzy models. These, are based on stability constraints transformable into LMIs to obtain the gains matrices. Among these different control laws we cite:

3.5.1 Parallel distributed compensation concept

The main idea of the PDC controller design is to derive each control rule from the corresponding rule of T-S fuzzy model so as to compensate it. The resulting overall fuzzy controller, which is nonlinear in general, is a fuzzy blending of each individual linear controller, knowing that the fuzzy controller shares the same fuzzy sets with the fuzzy model (3.6). Wang et al. (Wang *et al.* , 1996) used this concept to design fuzzy controllers to stabilize fuzzy systems. Figure 3.1 shows the concept of PDC controller (Tanaka & Sano, 1994).

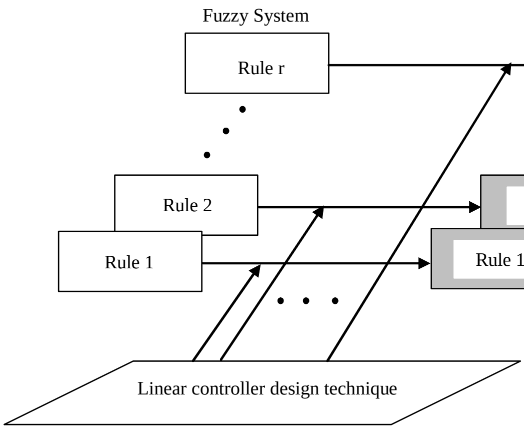

Figure 3.1: PDC controller design

For the fuzzy system (3.6), the following fuzzy controller via PDC is obtained (Tanaka & Wang, 2001, a):

$$\text{Rule } i : \text{IF } z_1(t) \text{ is } M_{i1} \text{ and ... and } z_p(t) \text{ is } M_{ip}$$

$$\text{THEN } u(t) = -F_i x(t), \; i = 1, 2, .., r \tag{3.7}$$

which has a state feedback controller in the consequent parts. The overall fuzzy controller is represented by

$$u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i x(t) \tag{3.8}$$

The PDC scheme that stabilizes the T-S fuzzy model was proposed by Wang et al. (Wang *et al.* , 1995),(Wang *et al.* , 1996), as a design framework comprising a control algorithm and a stability test using optimization involving LMI constraints. The goal is to find appropriated F_i gains that ensure the closed loop stability.

3.5.2 State feedback control

The control action is given by:

$$u(t) = -K_e x(t), K_e \in \mathbb{R}^{m \times n}$$

and the closed loop T-S fuzzy system is given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i - B_i K_e] x(t) \tag{3.9}$$

This control law allows a pole placement such that for any $x(t) \neq 0, x(t) \longrightarrow 0$ when $t \longrightarrow \infty$. However, the principle drawback is the performance limitation.

3.5.3 Fuzzy simultaneous stabilization (FSS)

This nonlinear state feedback control law was developed by Vermeirin (Vermeirin, 1998) and is based on Peterson's works (Petersen, 1987), concerning the simultaneous stabilization of MIMO linear models using a nonlinear state feedback control law. The FSS control law is given by:

$$u(x) = g_1(x) + g_2(x) \tag{3.10}$$

and the closed loop T-S fuzzy model is given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) - B_i(g_1(x) + g_2(x))) \tag{3.11}$$

The obtained stability conditions are more conservative than those of the PDC control. However, this approach consists the basis for other synthesis methods.

3.5.4 Compensation and division for fuzzy models (CDF)

This control law avoids the use of cross models. It requires a linear dependency between the input matrices and is given by:

$$u(t) = -\frac{\sum_i h_i(z(t)) k_i F_i x(t)}{\sum_{i=1} h_i(z(t)) k_i}, \; k_i > 0 \tag{3.12}$$

where F_i are the control gains. The closed loop T-S fuzzy model is given by (Guerra & Vermeirin, 1998):

$$\dot{x}(t) = \frac{\sum_i \sum_j h_i(z(t)) h_j(z(t)) k_j [A_i - B_i F_j]}{\sum_i h_i(z(t)) k_i} x(t)$$

Using the dependency property of this control law i.e. $B_i = K_i B$, the closed loop T-S fuzzy model becomes

$$\dot{x}(t) = \sum_i h_i(z(t))(A_i - B_i F_i) x(t) \tag{3.13}$$

the conservatism is reduced since there is only r LMIs instead of $r(r+1)/2$ LMIs for the PDC case.

3.6 Linear matrix inequalities (LMIs)

Linear Matrix Inequalities are the control version of the semi definite programs (SDP) that are convex problems, allowing the resolution of a great number of problems in relation with uncertain systems. A powerful and efficient polynomial-time interior-point algorithms were developed for linear programming by Karmakar in 1984 (Karmakar, 1984), then extended in 1988 by Nesterov and Nemirovskii, which developed interior-point methods that apply directly to linear matrix inequalities (Nestrov & Nemirovski, 1994). It was then recognized that LMIs can be solved with convex optimization on a computer and in 1995 Gahinet and Nemirovskii (Gahinet *et al.* , 1995) wrote a commercial Matlab package called the LMI Toolbox for Matlab. The advantage of SDP is the polynomial time of global minimum computation using the interior point methods (Nestrov & Nemirovski, 1994).

Definition 7 *(Boyd* et al. *, 1994) A linear matrix inequality is a matrix inequality of the form:*

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{3.14}$$

where $x(t) = [x_1(t), ..., x_m(t)]^T$ is the variable vector to find and $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i = 0, ..., m$ are given matrices. The inequality implies that $F(x)$ must be positive definite, i.e. all its eigenvalues are positive. The LMI (3.14) is a convex constraint on x , i.e. the set $\{x \mid F(x) > 0\}$ is convex, it can also gather several convex constraints. $F_1(x) > 0, F_2(x) > 0, ..., F_m(x) > 0$, in a diagonal bloc matrix:

$$\begin{bmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & F_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_m(x) \end{bmatrix} > 0 \tag{3.15}$$

3.6.1 Some standards LMI Problems

Among the most encountered convex optimization LMI problems, we cite:

LMI Problems:

Given a LMI $F(x) > 0$, the LMI problem is to find x^{feas} such that $F\left(x^{feas}\right) > 0$ or determine that the LMI is infeasible, which is a convex feasibility problem that can be solved by convex optimization algorithms such as interior-point methods. For example, the Lyapunov stability conditions given in section 3.4, will be expressed as an LMI problem where P is the variable (Boyd *et al.* , 1994), and this, is available for all the stability conditions encountered in this work.

Eigenvalue problem

The eigenvalue problem (EVP) is to minimize the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an LMI constraint (or determine that the constraint is infeasible), in other terms:

$$\begin{array}{cc} \textit{minimize} & \lambda \\ \textit{subject to} & \lambda I - A(x) > 0, \quad B(x) > 0 \end{array} \tag{3.16}$$

Generalized eigenvalue problem

The generalized eigenvalue problem (GEVP) is to minimize the maximum eigenvalue problem of a pair of matrices that depend affinely on a variable, subject to an LMI constraint. The general form of GEVP is:

$$\begin{array}{cc} \textit{minimize} & \lambda \\ \textit{subject to} & \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0 \end{array} \tag{3.17}$$

All these problems can be solved by different tools such as ellipsoid algorithms, simplex methods and interior-point methods. However, there exist some tools that facilitates the passage from a non convex formulation to a LMI, that is convex, among them:

Schur complement

Nonlinear (convex) inequalities are converted to LMI form using Schur complements. For the following LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \tag{3.18}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$ and $S(x)$ depend affinely on x is equivalent to

$$R(x) > 0, \; Q(x) - S(x) R(x)^{-1} S(x)^T > 0 \tag{3.19}$$

The lemma is also valid when changing the sign of the inequalities.

Another important property is the polytopic form of the T-S fuzzy systems

Polytopic form

A polytopic form is defined as follows: A set of matrices $\{A_1, A_2, ..., A_n\}$ is said to be polytopic if there exists a set of positive parameters such that (Zhao, 1995)

$$\forall\ 0 \leq \lambda_i \leq 1,\ \sum_{i=1}^{n} \lambda_i = 1,\ A = \sum_{i=1}^{n} \lambda_i A_i > 0$$

hence the matrices form a polytopic $\Lambda = C_o \{A_1, A_2, ..., A_n\}$, where C_o denotes the convex hull. The notion of convexity plays an important role since the stability analysis problems are represented in terms of convex optimization problems, what allows a reasonable computing time and finding a global minimum.

3.7 Quadratic stabilization of T-S fuzzy systems

The quadratic stabilization of T-S fuzzy models is not other than a state feedback stabilization design problem that can be stated as follows: given a plant described by a T-S model, find a PDC controller that quadratically stabilizes the closed loop system. The design variables in this problem are the gain matrices F_i $(1 \leq i \leq r)$. As stated previously in this chapter, the control design problem is to find the gains F_i such that the following closed loop system (3.22) is quadratically stable.

3.7.1 Stability conditions in closed loop

The overall T-S fuzzy system is given by:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right), \tag{3.20}$$

$$y(t) = \sum_{i=1}^{r} h_i(z(t))\, C_i x(t) \tag{3.21}$$

We note that equation (3.20) is a polytopic form of the fuzzy system. Hence, by substituting (3.8) in (3.20), we obtain the T-S closed loop fuzzy system as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))\, h_j(z(t)) \left[A_i - B_i F_j\right] x(t), \tag{3.22}$$

which can be rewritten as

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\, h_i(z(t))\, G_{ii} x(t) + 2 \sum_{i=1}^{r} \sum_{i<j} h_i(z(t))\, h_j(z(t)) \left\{ \frac{G_{ij} + G_{ji}}{2} \right\} x(t), \tag{3.23}$$

where $G_{ij} = A_i - B_i F_j$ and $G_{ii} = A_i - B_i F_i$.

Applying theorem 6 to the system defined by (3.23), we obtain the closed loop stability conditions given by the following theorem.

Theorem 8 *(Tanaka & Wang, 2001, a) The equilibrium of the continuous fuzzy control system described by (3.23) is globally asymptotically stable if there exists a common positive definite matrix P such that the following two conditions are satisfied:*

$$G_{ii}^T P + P G_{ii} < 0, \quad i = 1,\ ..., r, \tag{3.24}$$

$$\begin{aligned} \left\{\frac{G_{ij}+G_{ji}}{2}\right\}^T P + P\left\{\frac{G_{ij}+G_{ji}}{2}\right\} &\leq 0, \quad i = 1,\ ..., r, \\ i &< j \ s.t.\ h_i \cap h_j \neq \varnothing \end{aligned} \tag{3.25}$$

The proof of this theorem follows directly from theorem 6. We remark that condition (3.25) contributes to the conservatism reduction since, it is not necessary that all the sub-models are stable.

The common B matrix case

By considering $B_1 = B_2 = ... = B_r$, the stability condition of theorem 8 can be simplified as follows.

Corollary 9 *Assume that $B_1 = B_2 = ... = B_r$. The equilibrium of the fuzzy control system (3.23) is globally asymptotically stable if there exist a common positive matrix P satisfying (3.24).*

The stabilization of a feedback model containing a state feedback fuzzy controller has been extensively considered. The objective is to select F_i to stabilize the closed-loop system. The stability conditions corresponding to a quadratic Lyapunov function were derived by Tanaka and Sugeno in (Tanaka *et al.* , 1998). They give sufficient conditions for the quadratic stabilization by the following theorem:

Theorem 10 *(Tanaka & Wang, 2001, a) The fuzzy system (3.22) can be stabilized via the PDC controller (3.8) if there exists a common positive definite matrix X and M_i $(i = 1\ ...\ r)$ such that*

$$-XA_i^T - A_i X + M_i^T B_i^T + B_i M_i > 0, \tag{3.26}$$

$$\begin{aligned} &-XA_i^T - A_i X - XA_j^T - A_j X + M_j^T B_i^T + B_i M_j \\ &+M_i^T B_j^T + B_j M_i \geq 0, \quad \forall\ i < j\ s.t.\ h_i \cap h_j \neq \varnothing \end{aligned} \tag{3.27}$$

where

$$X = P^{-1},\ M_i = F_i X \tag{3.28}$$

See appendix B for LMIs transformations.

The feedback gains F_i and the common P are given by

$$P = X^{-1},\ F_i = M_i X^{-1} \tag{3.29}$$

whereas the single quadratic Lyapunov function is given by

$$V(x(t)) = x(t) X^{-1} x(t) \tag{3.30}$$

This approach requires to find a common positive definite matrix P for r sub-models, what makes it very conservative. An attempt to reduce the conservatism using the same Lyapunov function was given by Tanaka et al. (Tanaka *et al.* , 1998), who proposed relaxed stability conditions given by this theorem.

Theorem 11 *(Tanaka* et al. *, 1998)Assume that the number of rules that fire for all t is less than or equal to s, where $1 < s \leq r$. The equilibrium of the continuous fuzzy control system described by 3.23 is globally asymptotically stable if there exist a common positive definite matrix P and a common positive semidefinite matrix Q such that*

$$A_i^T P + P A_i - (s-1) Q < 0,\ i = 1, ..., r \tag{3.31}$$

$$\begin{aligned} \left\{\frac{G_{ij}+G_{ji}}{2}\right\}^T P + P\left\{\frac{G_{ij}+G_{ji}}{2}\right\} - Q &\leq 0,\ i = 1, ..., r, \\ \forall\ i &< j\ s.t.\ h_i \cap h_j \neq \varnothing \end{aligned} \tag{3.32}$$

where $s > 1$.

3.8 Design example: The Inverted pendulum

Consider now the problem of balancing and swing up an inverted pendulum. We recall the equations of motion already given in chapter 2 (Tanaka & Wang, 2001, a):

$$\begin{aligned} x_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{g\sin(x_1(t)) - amlx_2^2(t)\sin(2(x_1(t)))/2 - a\cos(x_1(t))u(t)}{4l/3 - aml\cos^2(x_1(t))}, \end{aligned} \tag{3.33}$$

For the simulations, the values of the parameters are $m = 2.0\ kg$, $M = 8.0\ kg$, $2l = 1.0\ m$. The control objective for this example is to balance the inverted pendulum for the approximate range $x_1 \in (-\pi/2, \pi/2)$ by using the quadratic stability approach and PDC controller. The system (3.33) is modeled by the following two fuzzy rules:

$$\text{Rule 1} \quad : \quad \text{IF } x_1(t) \text{ is about 0 THEN } \dot{x}(t) = A_1 x(t) + B_1 u(t) \tag{3.34}$$

$$\text{Rule 2} \quad : \quad \text{IF } x_1(t) \text{ is about } \pm\pi/2 \ (|x_1| < \pi/2) \text{ THEN } \dot{x}(t) = A_2 x(t) + B_2 u(t) \tag{3.35}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{4l/3-aml} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi\left(4l/3-aml\beta^2\right)} & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{bmatrix} \text{ and } \beta = \cos(88^\circ)$$

The PDC control laws are as follows:

$$\text{Rule 1} \quad : \quad \text{IF } x_1(t) \text{ is about 0 THEN } u(t) = -F_1 x(t) \tag{3.36}$$

$$\text{Rule 2} \quad : \quad \text{IF } x_1(t) \text{ is about } \pm\pi/2\,(|x_1| < \pi/2) \text{ THEN } u(t) = -F_2 x(t) \tag{3.37}$$

Hence, the control law that guarantees the stability of the fuzzy control system (fuzzy system + PDC controller) is given by:

$$u(t) = -h_1(x_1(t))\,F_1 x(t) - h_2(x_1(t))\,F_2 x(t) \tag{3.38}$$

where $h_1$and h_2 are the triangular membership functions of rules 1 and 2, respectively.

Applying then the PDC controller, the objective of balancing the pendulum is reached for initial conditions $x_1 \in (-\pi/2, \pi/2)$ and $x_2 = 0$. By resolving the LMI conditions (3.26) and (3.27), the values of P and F_i are:

$$P = \begin{bmatrix} 0.0160 & 0.0037 \\ 0.0037 & 0.0009 \end{bmatrix} > 0,$$

$$F_1 = \begin{bmatrix} -4347.9 & -981.7 \end{bmatrix}, \ F_2 = \begin{bmatrix} 10819 & 2465 \end{bmatrix}$$

Figure 3.2 shows the response of the pendulum system using fuzzy PDC controls for the initial conditions $x_1 = 30^\circ$, 60°, 85° and.$x_2 = 0$.

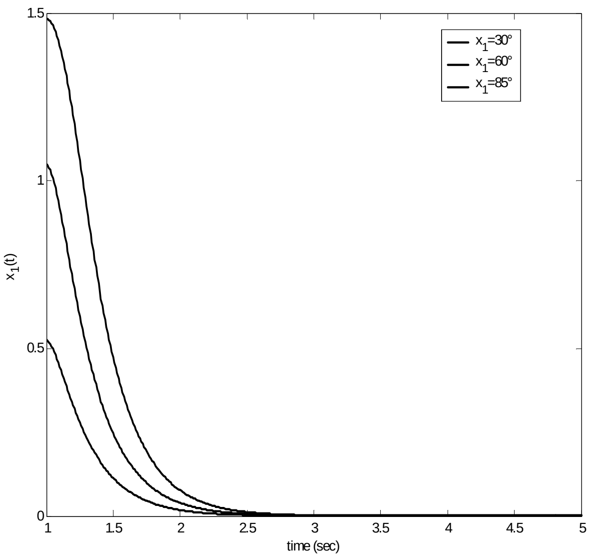

Figure 3.2: Angle response using two-rule fuzzy control for $x_1 = 30°$, $60°$, $85°$ and $x_2 = 0$.

To assess the effectiveness of the PDC controller, this one is applied to the original system (3.33), the results are also good as the figure 3.3 shows it for initial conditions $x_1 = 85°$, $15°$, $-85°$ and $x_2 = 0$. To test the robustness of this controller, some simulations are done by changing m from 2.0 to 4.0 kg (figure 3.4), M from 8.0 to 4.0 kg, (figure 3.5) and $2l$ from 1.0 to 0.5 m (figure 3.6) for different initial conditions. The obtained results are satisfactory knowing that robustness is not included in the controller design.

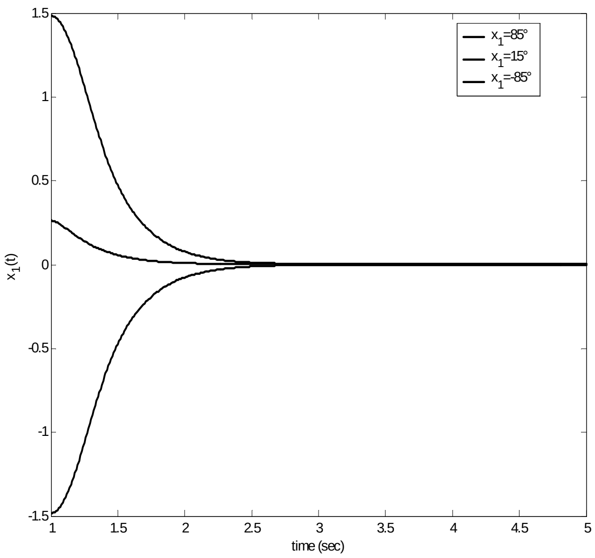

Figure 3.3: Angle response using two-rule fuzzy control for $x_1 = 85°$, $15°$, $-85°$ and $x_2 = 0$.

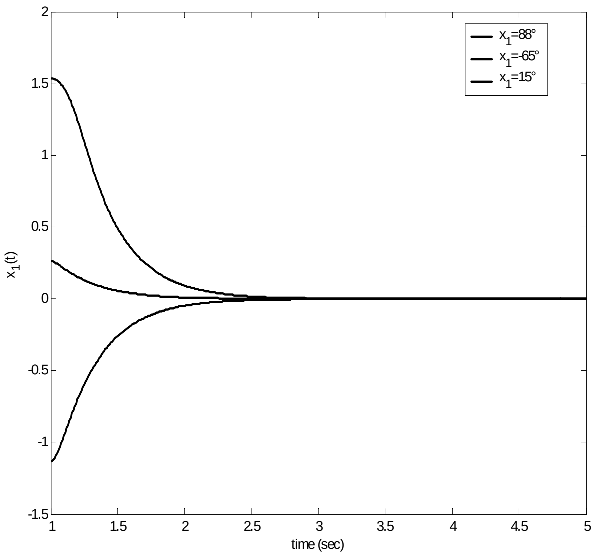

Figure 3.4: Angle response using two-rule fuzzy control and m changed from 2.0 to 4.0 kg.

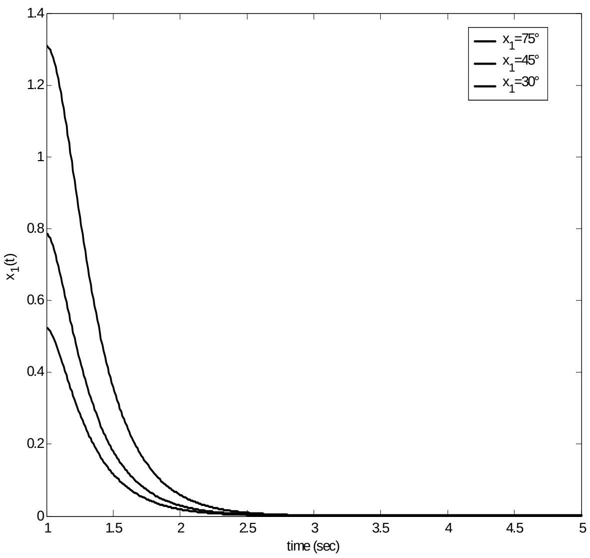

Figure 3.5: Angle response using two-rule fuzzy control and M changed from 8.0 to 4.0 kg.

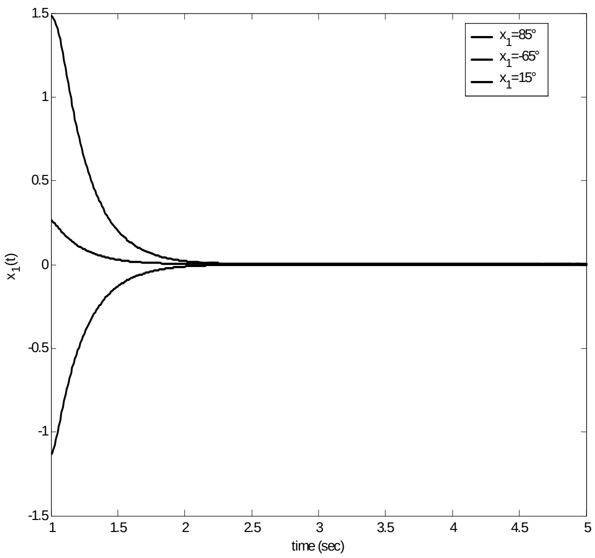

Figure 3.6: Angle response using two-rule fuzzy control and $2l$ changed from 1.0 to 0.5 m.

3.9 Conclusion

In this chapter, we presented the quadratic stability and stabilization of T-S fuzzy systems within the framework of the use of a PDC control law. The stability of the fuzzy system is based on the Lyapunov direct method consisting in finding a quadratic function $V(x(t)) = x^T(t)Px(t)$ whose derivative is negative. In this context, various theorems were introduced for the stability of fuzzy systems in open loop and in closed loop for stabilization. These, constitute feasible problems whose solutions can be found by LMIs tools, also detailed in this chapter. The proposed quadratic stability conditions are sufficient. However, they are conservative since the $h_i(z(t))$ are not taken into account ,it omits all the information contained in the membership functions, further the approach requires to find a common positive definite matrix P for r sub-models. Finally, an example is given to illustrate all these concepts combined in a framework entitled stable fuzzy controller design.

Chapter 4

Non-Quadratic Stability and Stabilization of Takagi-Sugeno Fuzzy Systems

4.1 Introduction

In this chapter, new stability conditions for the T-S fuzzy systems are proposed (Abdelmalek *et al.* , 2007), based on the use of multiple Lyapunov functions that have been discussed (Cao *et al.* , 1997),(Feng *et al.* , 1997),(Jadbabaie, 1999),(Chadli *et al.* , 2000),(Tanaka *et al.* , 2001, c),(Hadjili, 2002) due to their properties of conservatism reduction. The proposed stability conditions are derived based on PDC controller. We use a fuzzy Lyapunov function since it is smooth contrary to the piecewise Lyapunov function thus avoiding the boundary condition problem. Hence we obtain new conditions, shown to be less conservative, that stabilize continuous T-S fuzzy systems including those that do not admit a quadratic stabilization. The proposed approach is based on two assumptions. The first one relies on the existence of a proportionality relation between multiple quadratic Lyapunov functions, and the second one considers an upper bound for the time derivative of the premise membership function as assumed by Tanaka et al. (Tanaka *et al.* , 2001, b),(Tanaka *et al.* , 2001, c),(Tanaka *et al.* , 2001, d), (Tanaka *et al.* , 2003). Simulation examples demonstrate the effectiveness of the proposed approach.

4.2 Non-quadratic stability of T-S fuzzy models

Due to their property of conservatism reduction, we define a fuzzy Lyapunov function and consider stability conditions via the fuzzy Lyapunov function. The candidate

Lyapunov function

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\, x^T(t)\, P_i x(t) \tag{4.1}$$

satisfies

V is C^1,

$V(0) \neq 0$ and $V(x(t)) > 0$ for $x(t) \neq 0$,

$\|x(t)\| \longrightarrow \infty \Rightarrow V(x(t)) \longrightarrow \infty$

Definition 12 *(Tanaka & Sugeno, 1992) Equation (4.1) is said to be a fuzzy Lyapunov function for the T-S fuzzy system if the time derivative of $V(x(t))$ is always negative at $x(t) \neq 0$, where P_i is a positive definite matrix.*

This fuzzy Lyapunov function is defined (Tanaka *et al.* , 2003) for studying the stability and stabilization of the following continuous T-S fuzzy system.

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\,(A_i x(t) + B_i u(t)) \tag{4.2}$$

Applying the PDC controller

$$u(t) = -\sum_{i=1}^{r} h_i(z(t))\, F_i x(t) \tag{4.3}$$

the T-S closed loop fuzzy system is:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\,[A_i - B_i F_j]\, x(t) \tag{4.4}$$

4.3 New stabilization approach

In this section, and based on the fuzzy Lyapunov function, we propose an approach that gives less conservative stability conditions (Abdelmalek *et al.* , 2007) The Key assumptions are as follows:

> **Assumption 1:** The time derivative of the premise membership function are upper bounded such that: $\left|\dot{h}_i(z(t))\right| \leq \phi_i$ for $i = 1, .., r$, where ϕ_i, $i = 1, ..., r$ are given positive constants.
>
> **Assumption 2:** The local quadratic Lyapunov functions $x^T(t)\, P_i x(t)$, $i = 1, ..., r$ are proportionally related such that: $P_j = \alpha_{ij} P_i$ for $i, j = 1, ..., r$, where $\alpha_{ij} \neq 1$ and $\alpha_{ij} > 0$ for $i \neq j$, and $\alpha_{ij} = 1$ for $i = j$.

Theorem 13 *Under assumptions 1-2, the continuous fuzzy system (4.4) can be stabilized via the PDC fuzzy controller (4.3) if there exist* ϕ_ρ, α_{ij} *for* $i, j, \rho = 1, ..., r$, *positive definite matrices* $P_1, P_2, ..., P_r$ *and matrices* $F_1, F_2, ..., F_r$ *such that*

$$P_i > 0, \qquad i = 1, 2, .., r \tag{4.5}$$

$$\sum_{\rho=1}^{r} \phi_\rho P_\rho + \left(G_{jj}^T P_i + P_i G_{jj}\right) < 0, \qquad i, j = 1, 2, .., r \tag{4.6}$$

$$\left\{\frac{G_{jk} + G_{kj}}{2}\right\}^T P_i + P_i \left\{\frac{G_{jk} + G_{kj}}{2}\right\} < 0, \quad \forall\ i, j, k \in \{1, 2, .., r\}\ \ s.t.\ \ j < k \tag{4.7}$$

where $G_{jk} = A_j - B_j F_k$ *and* $G_{jj} = A_j - B_j F_j$.

Proof. The candidate Lyapunov function is defined by

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\, x^T(t)\, P_i x(t) \tag{4.8}$$

The time derivative of $V(x(t))$ is calculated as

$$\begin{aligned} \dot{V}(x(t)) = {} & \dot{x}^T(t) \left(\sum_{i=1}^{r} h_i(z(t))\, P_i\right) x(t) \\ & + x^T(t) \left(\sum_{\rho=1}^{r} \dot{h}_\rho(z(t))\, P_\rho\right) x(t) \\ & + x^T(t) \left(\sum_{i=1}^{r} h_i(z(t))\, P_i\right) \dot{x}(t) \end{aligned} \tag{4.9}$$

By substituting the following

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\, h_i(z(t))\, G_{ii} x(t) + 2 \sum_{i=1}^{r} \sum_{i<j} h_i(z(t))\, h_j(z(t)) \left\{\frac{G_{ij} + G_{ji}}{2}\right\} x(t)$$

in (4.9), we obtain

$$\begin{aligned} \dot{V}(x(t)) = x^T(t) \Bigg[& \sum_{j=1}^{r} \sum_{i=1}^{r} h_j(z(t))\, h_j(z(t))\, h_i(z(t))\, G_{jj}^T P_i \\ & + \sum_{j=1}^{r} \sum_{j<k} \sum_{i=1}^{r} h_j(z(t))\, h_k(z(t))\, h_i(z(t)) \left(\frac{G_{jk} + G_{kj}}{2}\right)^T P_i \\ & + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t))\, P_\rho + \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, h_j(z(t))\, P_i G_{jj} \\ & + \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t))\, P_i \left(\frac{G_{jk} + G_{kj}}{2}\right) \Bigg] x(t) \end{aligned} \tag{4.10}$$

Further, (4.9) can be arranged as:

$$\dot{V}(x(t)) = x^T(t)\left[\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, h_j(z(t))\left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t))\, P_\rho \right.$$
$$\left. + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t))\left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i\left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\right] x(t). \tag{4.11}$$

Under the assumption $\left|\dot{h}_\rho(z(t))\right| \leq \phi_\rho$, (4.11) can be rewritten as follows:

$$\dot{V}(x(t)) \leq x^T(t)\left[\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, h_j(z(t))\left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r} \phi_\rho P_\rho \right.$$
$$\left. + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t))\left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i\left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\right] x(t) \tag{4.12}$$

If equations (4.5)-(4.7) hold, the time derivative of the fuzzy Lyapunov function is negative. So we have

$$\dot{V}(x(t)) \leq x^T(t)\left[\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j^2(z(t))\left(\left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r} \phi_\rho P_\rho\right)\right.$$
$$\left. + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t))\left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i\left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\right] x(t) < 0,$$

and the closed loop fuzzy system (4.4) is stable. ■

4.3.1 Constraints on the time derivative of the premise membership functions

Conditions of theorem 13 were derived by including an assumption on the time derivative of the premise membership functions

$$\left|\dot{h}_\rho(z(t))\right| \leq \phi_\rho \quad for\ \rho = 1, ..., r \tag{4.13}$$

In this subsection, the constraint imposed on the time derivative of the premise membership functions and hence on the derivative of the premise variables, (i.e. the speed of the state variables for the case of $z(t) = x(t)$), is transformed into LMIs of theorem 14 that are solved simultaneously with those of theorem 13 to stabilize the T-S fuzzy systems. The new LMIs that support assumption 1, allows to increase the performance by limiting the displacement rate in the polytope, implying a facility to find the Lyapunov functions and thus a faster stabilization.

Theorem 14 *Assume that $x(0)$ and $z(0)$ are known. The assumption (4.13) holds if there exist positive definite matrices $P_1, P_2, \ldots, P_r$ and matrices $F_1, F_2, \ldots, F_r$ satisfying*

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & P_i^{-1} \end{bmatrix} \geq 0,\ for\ i = 1,..,r \tag{4.14}$$

$$\begin{bmatrix} \phi_\rho P_i & W_{ij\rho\ell}^T \\ W_{ij\rho\ell} & \phi_\rho I \end{bmatrix} \geq 0,\ \forall\ i,j,\rho \in \{1,2,..,r\},\ \forall \ell \tag{4.15}$$

where $W_{ij\rho\ell} = \xi_{\rho\ell}(A_i - B_i F_j)$. The selection of $\xi_{\rho\ell}$ is obtained from $\dot{h}_i(z(t))$ using a simple procedure (see appendix C) given in (Tanaka et al. *, 2001, d). However, it is to be noted that the conditions of this theorem are initial states dependent, so the initial conditions should be known and for different initial states, we need to solve the LMIs again.*

Proof. From (4.13) and for $z(t) = x(t)$ we have

$$\left|\dot{h}_\rho(z(t))\right| = \left|\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right| \leq \phi_\rho \tag{4.16}$$

we assume also that

$$\frac{\partial h_\rho(z(t))}{\partial x(t)} = \sum_{\ell=1}^{s} \upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell} \tag{4.17}$$

where $\upsilon_{\rho\ell}(z(t)) \geq 0$ and $\sum_{\ell=1}^{s} \upsilon_{\rho\ell}(z(t)) = 1$. Using (4.17) we obtain LMIs that satisfy assumption (4.16).

From (4.16) we have

$$\left(\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right)^T \left(\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right) \leq \phi_\rho^2 \tag{4.18}$$

By replacing (4.4) in (4.18) we obtain

$$\begin{aligned}&\left[\left(\sum_{\ell=1}^{s} \upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\,[A_i - B_i F_j]\,x(t)\right\}\right)^T\right. \\ &\left.\times\left(\sum_{\ell=1}^{s} \upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\,[A_i - B_i F_j]\,x(t)\right\}\right)\right] \\ &\leq \phi_\rho^2\end{aligned} \tag{4.19}$$

Dividing by ϕ_ρ^2 we obtain

$$\begin{aligned}&\frac{1}{\phi_\rho^2}x^T(t)\left[\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]^T\right\}\right)\right.\\&\left.\times\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]\right\}\right)\right]x(t)\\&\leq 1\end{aligned} \tag{4.20}$$

We assume that for the fuzzy Lyapunov function (4.1), the inequality (4.21) holds (Tanaka & Wang, 2001, a),(Bernal & Hušek, 2005):

$$V(x(t))\leq V(x(0))\leq 1, t\geq 0 \tag{4.21}$$

i.e.

$$\sum_{i=1}^{r}h_i(z(t))\,x^T(t)\,P_ix(t)\leq\sum_{i=1}^{r}h_i(z(0))\,x^T(0)\,P_ix(0)\leq 1 \tag{4.22}$$

then we have

$$1-\sum_{i=1}^{r}h_i(z(0))\,x^T(0)\,P_ix(0)\geq 0 \tag{4.23}$$

and

$$1-x^T(0)\left(\sum_{i=1}^{r}h_i(z(0))\,P_i\right)x(0)\geq 0 \tag{4.24}$$

then we have

$$1-\sum_{i=1}^{r}h_i(z(0))\,x^T(0)\,P_ix(0)\geq 0 \tag{4.25}$$

and

$$1-x^T(0)\left(\sum_{i=1}^{r}h_i(z(0))\,P_i\right)x(0)\geq 0 \tag{4.26}$$

which is expressed via LMIs using the Schur complement as follows:

$$\begin{bmatrix}1 & x^T(0)\\ x(0) & \left(\sum_{i=1}^{r}h_i(z(0))\,P_i\right)^{-1}\end{bmatrix}\geq 0 \tag{4.27}$$

which is implied by

$$\begin{bmatrix}1 & x^T(0)\\ x(0) & P_i^{-1}\end{bmatrix}\geq 0,\ for\ i=1,..,r$$

that leads to the LMI condition (4.14).

On the other hand, by considering inequalities (4.20) and (4.22), inequality (4.16) holds if

$$\frac{1}{\phi_\rho^2}\left[\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}\left(z\left(t\right)\right)\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z\left(t\right)\right)h_j\left(z\left(t\right)\right)\left[A_i-B_iF_j\right]^T\right\}\right)\right.$$
$$\left.\times\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}\left(z\left(t\right)\right)\xi_{\rho\ell}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z\left(t\right)\right)h_j\left(z\left(t\right)\right)\left[A_i-B_iF_j\right]\right\}\right)\right]-\sum_{i=1}^{r}h_i\left(z\left(t\right)\right)P_i$$
$$\leq 0 \tag{4.28}$$

which is equivalent to

$$\left[\begin{array}{cc}\phi_\rho\sum_{i=1}^{r}h_i\left(z\left(t\right)\right)P_i & \left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}\left(z\left(t\right)\right)\xi_{\rho\ell}Q^T\right)\\ \left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}\left(z\left(t\right)\right)\xi_{\rho\ell}Q\right) & \phi_\rho I\end{array}\right]\geq 0 \tag{4.29}$$

where $Q=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z\left(t\right)\right)h_j\left(z\left(t\right)\right)\left[A_i-B_iF_j\right]$. This leads to the LMI condition (4.15)

$$\left[\begin{array}{cc}\phi_\rho P_i & W_{ij\rho\ell}^T\\ W_{ij\rho\ell} & \phi_\rho I\end{array}\right]\geq 0,\ \forall i,j,\rho\in\{1,2,..,r\}\ \forall\ell$$

where $W_{ij\rho\ell}=\xi_{\rho\ell}\left(A_i-B_iF_j\right)$. ■

4.3.2 Stable fuzzy controller design

We are interested in this part by non-quadratic stabilization of T-S fuzzy models by using PDC laws. The fuzzy controller design is to determine the local feedback gains F_i for the closed loop T-S fuzzy system (4.4). We define $X_i=P_i^{-1}$, $F_i=M_iX_i^{-1}$, $X_i=\alpha_{ij}X_j$ for $i,j=1,...,r$, where $\alpha_{ij}\neq 1$ and $\alpha_{ij}>0$ for $i\neq j$, and $\alpha_{ij}=1$ for $i=j$. By giving $\phi_\rho>0$ and α_{ij} for $i,j,\rho=1,...,r$, we obtain the following LMIs conditions that constitute a stable fuzzy controller design problem:

$$X_i>0,\ i=1,2,..,r \tag{4.30}$$

$$\sum_{\rho=1}^{r}\phi_\rho X_\rho+X_iA_j^T-\alpha_{ij}M_j^TB_j^T+A_jX_i-\alpha_{ij}B_jM_j<0,\ i,j=1,2,..,r \tag{4.31}$$

$$X_iA_j^T-\alpha_{ik}M_k^TB_j^T+X_iA_k^T-\alpha_{ij}M_j^TB_k^T+A_jX_i-\alpha_{ik}B_jM_k+A_kX_i-\alpha_{ij}B_kM_j<0$$
$$s.t.\ j<k,\ \forall\ i,j,k,\in\{1,2,...,r\}$$

$$\left[\begin{array}{cc}1 & x^T\left(0\right)\\ x\left(0\right) & X_i\end{array}\right]\geq 0,\ for\ i=1,...,r \tag{4.32}$$

$$\left[\begin{array}{cc} \phi_\rho X_i & W_{ij\rho\ell}^T \\ W_{ij\rho\ell} & \phi_\rho I \end{array}\right] \geq 0,\ \forall\ i,j,\rho \in \{1,2,...,r\}\,,\ \forall\ell \tag{4.33}$$

where $W_{ij\rho\ell} = \xi_{\rho\ell}\left(A_i X_i - \alpha_{ij} B_i M_j\right)$.

It is to be noted that from $X_i = \alpha_{ij} X_j$, we have $X_j = (1/\alpha_{ij})\, X_i = \alpha_{ji} X_i$ so $\alpha_{ij} = 1/\alpha_{ji}\ \forall\ i,j \in \{1,2,...,r\}$, hence according to our proposal and for a given i and j, the following relation is used $\alpha_{ij}\alpha_{ji} = 1$. Coefficients α_{ij} and ϕ_ρ for $i,j,\rho = 1,2,...,r$ and $i \neq j$, can be chosen heuristically according to the considered application. In particular, the $\phi_\rho s$ are chosen in such way to obtain a fast switching among IF-THEN rules in order to keep the speed of response for a closed loop system (Tanaka *et al.* , 2001, d).

4.4 Design examples

This part presents four different examples that illustrate the effectiveness of the new non-quadratic stabilization conditions that we propose in this chapter (Abdelmalek *et al.* , 2007).

4.4.1 Example 1

Consider the following fuzzy system (Tanaka *et al.* , 2001, b):

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) \tag{4.34}$$

$$h_1(x_1(t)) = \frac{1+\sin x_1(t)}{2}, h_2(x_1(t)) = \frac{1-\sin x_1(t)}{2}$$

$$A_1 = \left[\begin{array}{cc} -5 & -4 \\ -1 & -2 \end{array}\right], A_2 = \left[\begin{array}{cc} -2 & -4 \\ 20 & -2 \end{array}\right]$$

$$B_1 = \left[\begin{array}{c} 0 \\ 10 \end{array}\right], B_2 = \left[\begin{array}{c} 0 \\ 3 \end{array}\right]$$

For this fuzzy system that admits a quadratic stabilization it is assumed that $|x_1(t)| \leq \frac{\pi}{2}$ and $|x_2(t)| \leq \frac{\pi}{2}$. By taking

$$\begin{aligned} \phi_1 &= \phi_2 = 0.5 \\ \alpha_{12} &= 0.2,\ \alpha_{21} = 1/\alpha_{12} \\ \xi_{11} &= 0,\ \xi_{12} = 0.5,\ \xi_{21} = -0.5,\ \xi_{22} = 0 \end{aligned}$$

we obtain the following P_1, P_2, F_1 and F_2, that depend on initial conditions and satisfy the LMIs given in theorems 13 and 14 simultaneously:

$$\begin{aligned} P_1 &= \begin{bmatrix} 8.2039 & 1.0367 \\ 1.0367 & 3.0338 \end{bmatrix} > 0,\ P_2 = \begin{bmatrix} 30.5563 & -6.3970 \\ -6.3970 & 4.7558 \end{bmatrix} > 0 \\ F_1 &= \begin{bmatrix} 0.0262 & 0.1232 \end{bmatrix},\ F_2 = \begin{bmatrix} -3.4925 & 1.9967 \end{bmatrix} \end{aligned}$$

The new PDC fuzzy controller design conditions has feasible solutions for different initial conditions and hence stabilizes the system. Figure 4.1 shows, respectively, the states evolution and the control input for the initial values $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. It also shows that the conservatism reduction leads us to very interesting results with a fast convergence for the stabilization of the T-S fuzzy system comparing to those obtained in (Tanaka *et al.* , 2001, b).

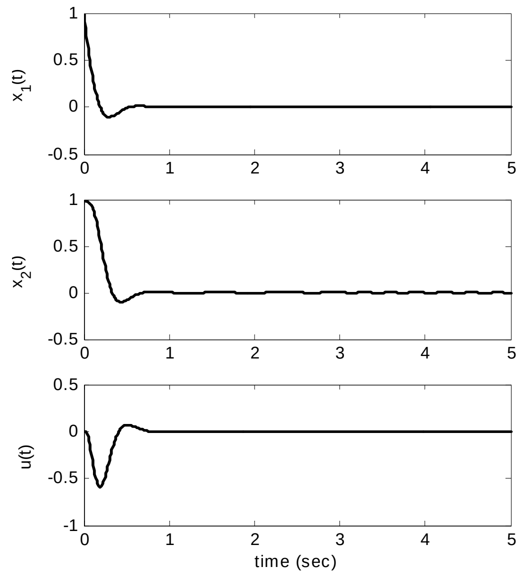

Figure 4.1: Example 1 performances.

4.4.2 Example 2

Here, another example which does not admit a single Lyapunov function (Morère, 2001).

$$h_1(x_1(t)) = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1} x_1(t)\right], h_2(x_1(t)) = \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1} x_1(t)\right]$$

$$A_1 = \begin{bmatrix} 0.1000 & -1.0000 \\ -0.2500 & 1.0000 \end{bmatrix}, A_2 = \begin{bmatrix} 1.0000 & 0.5000 \\ 0.7500 & 2.0000 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.6500 \\ -0.2000 \end{bmatrix}, B_2 = \begin{bmatrix} -1.0000 \\ -0.0500 \end{bmatrix}$$

The design of a state-feedback controller using a common Lyapunov function is not possible since the corresponding LMI problem is infeasible. However, if we consider local Lyapunov functions, the LMI problem (4.30)-(4.33) is feasible. The proposed approach gives feasible solutions for different initial conditions, and thus stabilizes the T-S closed loop system. For

$$\begin{aligned} \phi_1 &= \phi_2 = 5 \\ \alpha_{12} &= 1.5,\ \alpha_{21} = 1/\alpha_{12} \\ \xi_{11} &= 0.25,\ \xi_{12} = 0.75,\ \xi_{21} = 0.25,\ \xi_{22} = 0.75 \end{aligned}$$

we obtain the following P_1, P_2, F_1 and F_2 :

$$P_1 = \begin{bmatrix} 12.1789 & -104.4753 \\ -104.4753 & 997.4141 \end{bmatrix} > 0,\ P_2 = \begin{bmatrix} 12.3823 & -103.6178 \\ -103.6178 & 989.7426 \end{bmatrix} > 0$$
$$F_1 = \begin{bmatrix} 14.1362 & -211.3544 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.3676 & -72.8607 \end{bmatrix}$$

Figure 4.2 shows, respectively, the system's states and control evolution, for the initial values $x(0) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$.

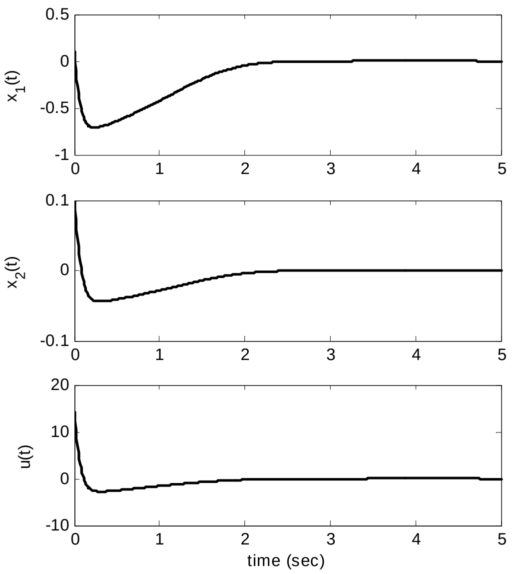

Figure 4.2: Example 2 performances.

4.4.3 Example 3: The *Inverted Pendulum*

Consider now the problem of balancing and swing-up an inverted pendulum using the approach that we propose in this chapter. The equations of motion are given in chapter 3 by (3.33) (Tanaka & Wang, 2001, a). The control objective for this example is to balance the inverted pendulum for the approximate range $x_1 \in (-\pi/2, \pi/2)$. The nonlinear system is modeled by two fuzzy rules (3.34) and (3.35), where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{2g}{4l/3-aml} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi\left(4l/3-aml\beta^2\right)} & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{bmatrix} \textit{ and } \beta = \cos\left(88^\circ\right). \end{aligned}$$

The PDC control laws are given by (3.36) and (3.37). The control law that guarantees the stability of the closed loop fuzzy system is given by:

$$u(t) = -h_1(x_1(t))\, F_1 x(t) - h_2(x_1(t))\, F_2 x(t)$$

where $h_1(x_1(t))$ and $h_2(x_1(t))$ are the triangular membership functions of rules 1 and 2, respectively.

Applying the proposed approach, the objective of balancing and stabilizing the pendulum is reached with success for different initial conditions of $x_1 \in (-\pi/2, \pi/2)$ and $x_2 = 0$. For

$$\begin{aligned} \phi_1 &= \phi_2 = \pi/1.5 \\ \alpha_{12} &= 1.3,\ \alpha_{21} = 1/\alpha_{12} \\ \xi_{11} &= -2/\pi,\ \xi_{12} = 2/\pi,\ \xi_{21} = -2/\pi,\ \xi_{22} = 2/\pi \end{aligned}$$

we obtain the following P_1, P_2, F_1 and F_2 for each initial condition.

For $x(0) = \begin{bmatrix} \pi/6 & 0 \end{bmatrix}^T$:

$$\begin{aligned} P_1 &= \begin{bmatrix} 57.7603 & 23.2068 \\ 23.2068 & 10.3697 \end{bmatrix} > 0, P_2 = \begin{bmatrix} 58.1998 & 17.5082 \\ 17.5082 & 6.1428 \end{bmatrix} > 0 \\ F_1 &= \begin{bmatrix} -630.7446 & -164.6591 \end{bmatrix},\ F_2 = 10^{-3} \begin{bmatrix} -1.2396 & -0.2958 \end{bmatrix} \end{aligned}$$

For $x(0) = \begin{bmatrix} \pi/4 & 0 \end{bmatrix}^T$:

$$\begin{aligned} P_1 &= \begin{bmatrix} 32.0668 & 13.1229 \\ 13.1229 & 6.4541 \end{bmatrix} > 0,\ P_2 = \begin{bmatrix} 39.1987 & 11.4436 \\ 11.4436 & 4.1936 \end{bmatrix} > 0 \\ F_1 &= \begin{bmatrix} -530.6214 & -127.4777 \end{bmatrix},\ F_2 = 10^{3} \begin{bmatrix} -1.0859 & -0.2427 \end{bmatrix} \end{aligned}$$

For $x(0) = \begin{bmatrix} \pi/3 & 0 \end{bmatrix}^T$:

$$\begin{aligned} P_1 &= \begin{bmatrix} 27.3149 & 10.8202 \\ 10.8202 & 5.6005 \end{bmatrix} > 0, \; P_2 = \begin{bmatrix} 51.3551 & 14.8473 \\ 14.8473 & 5.3225 \end{bmatrix} > 0 \\ F_1 &= \begin{bmatrix} -502.4650 & -115.6213 \end{bmatrix}, \; F_2 = 10^3 \begin{bmatrix} -1.3235 & -0.3102 \end{bmatrix} \end{aligned}$$

Figure 4.3 shows, respectively, the inverted pendulum position, velocity and control force evolution, for different initial conditions.

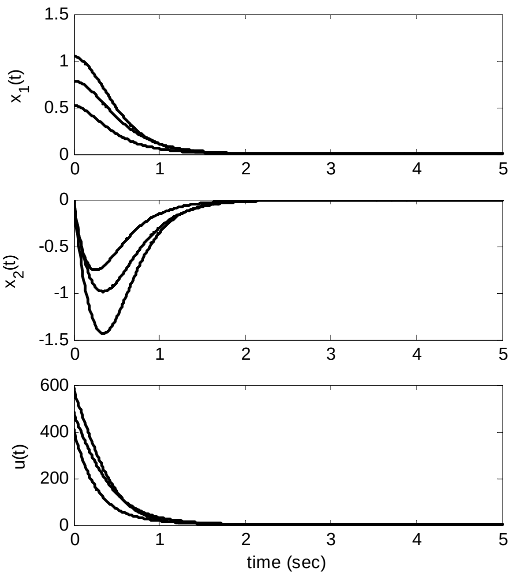

Figure 4.3: Example 3 performances.

4.4.4 Example 4: *Two-link robot manipulator*

To show the effectiveness of the proposed approach, we apply it to a more complicated system: a two-link robot manipulator (Tsen *et al.* , 2001, a). The dynamic equation of the two-link robot system is given as follows:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{4.35}$$

where

$$M(q) = \begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix}$$

$$C(q, \dot{q}) = m_2 l_1 l_2 (c_1 s_2 + s_1 c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} -(m_1 + m_2) l_1 g s_1 \\ -m_2 l_2 g s_2 \end{bmatrix}$$

and $q = [q_1, q_2]^T$, q_1, q_2 are generalized coordinates, $M(q)$ is the inertia matrix , $C\left(q, \dot{q}\right)$ includes coriolis, centripetal forces, and $G(q)$ is the gravitational force. The different parameters are: links masses m_1, $m_2\,(kg)$, links lengths l_1, $l_2\,(m)$, angular position q_1, $q_2\,(rad)$, applied torques $\tau = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}^T (N.m)$, the acceleration due to gravity $g = 9.8\,(m/s^2)$, and short-hand notation $s_1 = \sin(q_1)$, $s_2 = \sin(q_2)$, $c_1 = \cos(q_1)$, and $c_2 = \cos(q_2)$. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and $x_4 = \dot{q}_2$. The state space representation is given by:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_1(x) + g_{11}(x)\tau_1 + g_{12}\tau_2, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= f_2(x) + g_{21}(x)\tau_1 + g_{22}\tau_2, \\ y_1 &= x_1, \\ y_2 &= x_3 \end{aligned} \tag{4.36}$$

For more details concerning $f_1(x)$, $f_2(x)$, $g_{11}(x)$, g_{12}, $g_{21}(x)$ and g_{22}, see (Tsen *et al.* , 2001).

The objective is the fuzzy stabilization of the two-link robot using the non-quadratic approach. The links masses are $m_1 = 1\,(kg)$, $m_2 = 1\,(kg)$, the links lengths are $l_1 = 1\,(m)$, $l_2 = 1\,(m)$ and angular positions q_1, q_2 are constrained within $[-(\pi/2), (\pi/2)]$. The T-S fuzzy model for the system (4.36) is given by the following 9 rules whose membership functions are of triangular form (Tsen *et al.* , 2001):

$$\text{Rule 1}: \begin{cases} \text{IF } x_1(t) \text{ is about } -\pi/2 \text{ and } x_3(t) \text{ is about } \pi/2 \\ \text{THEN } \dot{x}(t) = A_1 x(t) + B_1 u(t),\ y = C_1 x(t) \end{cases}$$

$$\text{Rule 2}: \begin{cases} \text{IF } x_1(t) \text{ is about -}\pi/2 \text{ and } x_3(t) \text{ is about } 0 \\ \text{THEN} \dot{x}(t) = A_2 x(t) + B_2 u(t),\ y = C_2 x(t) \end{cases}$$

$$\text{Rule 3}: \begin{cases} \text{IF } x_1(t) \text{ is about } -\pi/2 \text{ and } x_3(t) \text{ is about } -\pi/2 \\ \text{THEN } \dot{x}(t) = A_3 x(t) + B_3 u(t),\ y = C_3 x(t) \end{cases}$$

$$\text{Rule 4}: \begin{cases} \text{IF } x_1(t) \text{ is about } 0 \text{ and } x_3(t) \text{ is about } -\pi/2 \\ \text{THEN } \dot{x}(t) = A_4 x(t) + B_4 u(t),\ y = C_4 x(t) \end{cases}$$

$$\text{Rule 5}: \begin{cases} \text{IF } x_1(t) \text{ is about } 0 \text{ and } x_3(t) \text{ is about } 0 \\ \text{THEN } \dot{x}(t) = A_5 x(t) + B_5 u(t),\ y = C_5 \end{cases}$$

$$\text{Rule 6}: \begin{cases} \text{IF } x_1(t) \text{ is about } 0 \text{ and } x_3(t) \text{ is about } \pi/2 \\ \text{THEN } \dot{x}(t) = A_6 x(t) + B_6 u(t),\ y = C_6 x(t) \end{cases}$$

$$\text{Rule 7}: \begin{cases} \text{IF } x_1(t) \text{ is about } \pi/2 \text{ and } x_3(t) \text{ is about } -\pi/2 \\ \text{THEN } \dot{x}(t) = A_7 x(t) + B_7 u(t),\ y = C_7 x(t) \end{cases}$$

$$\text{Rule 8}: \begin{cases} \text{IF } x_1(t) \text{ is about } \pi/2 \text{ and } x_3(t) \text{ is about } 0 \\ \text{THEN } \dot{x}(t) = A_8 x(t) + B_8 u(t),\ y = C_8 x(t) \end{cases}$$

$$\text{Rule 9}: \begin{cases} \text{IF } x_1(t) \text{ is about } \pi/2 \text{ and } x_3(t) \text{ is about } \pi/2 \\ \text{THEN } \dot{x}(t) = A_9 x(t) + B_9 u(t),\ y = C_9 x(t) \end{cases}$$

where $x = [x_1, x_2, x_3, x_4]^T$, $u = [\tau_1, \tau_2]^T$, and the local models matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -8.4\times10^{-6} \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 6.2\times10^{-6} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.0011 & 0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ 3.5436 & 0.0313 & 2.5611 & 1.14\times10^{-5} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.0030 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ 9.1041 & 0.0158 & -1.0574 & -3.2\times10^{-5} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.2427 & 0.0002 \\ 0 & 0 & 0 & 1 \\ -3.1873 & -0.0306 & 5.1911 & -1.8\times10^{-6} \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0.0 & -1.8145 & 0.0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0.0 & 9.1638 & 0.0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.0010 & 1.6870 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & 1.1\times10^{-5} \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1206 & -0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & -0.0193 & -1.0119 & 4.4 \times 10^{-5} \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & 0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.4290 & -0.0305 & 2.9832 & 1.9 \times 10^{-5} \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2933 & -0.0009 & -0.2188 & -1.2 \times 10^{-5} \\ 0 & 0 & 0 & 1 \\ -7.4649 & 0.0024 & 3.2693 & 9.2 \times 10^{-6} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For

$\phi_1 = 1$	$\phi_2 = 1.5$	$\phi_3 = 1.2$	$\phi_4 = 2$	$\phi_5 = 1.6$
$\phi_6 = 1.8$	$\phi_7 = 2.5$	$\phi_8 = 2.2$	$\phi_9 = 2.4$	

$\alpha_{12} = 1.2$	$\alpha_{13} = 0.66$	$\alpha_{14} = 0.9$	$\alpha_{15} = 0.8$	$\alpha_{16} = 0.7$
$\alpha_{17} = 1.5$	$\alpha_{18} = 2$	$\alpha_{19} = 1.1$	$\alpha_{23} = 1.6$	$\alpha_{24} = 1.18$
$\alpha_{25} = 1.05$	$\alpha_{26} = 1.9$	$\alpha_{27} = 1.99$	$\alpha_{28} = 0.99$	$\alpha_{29} = 0.77$
$\alpha_{34} = 0.88$	$\alpha_{35} = 1.33$	$\alpha_{36} = 1.4$	$\alpha_{37} = 1.7$	$\alpha_{38} = 1.66$
$\alpha_{39} = 1.56$	$\alpha_{45} = 1.48$	$\alpha_{46} = 1.39$	$\alpha_{47} = 1.69$	$\alpha_{48} = 1.11$
$\alpha_{49} = 1.88$	$\alpha_{56} = 2.1$	$\alpha_{57} = 2.2$	$\alpha_{58} = 1.61$	$\alpha_{59} = 1.23$
$\alpha_{67} = 2.11$	$\alpha_{68} = 2.2$	$\alpha_{69} = 1.52$	$\alpha_{78} = 0.78$	$\alpha_{79} = 0.98$
$\alpha_{89} = 0.82$				

$\xi_{11} = 0$	$\xi_{12} = 2/\pi$	$\xi_{21} = 0$	$\xi_{22} = 2/\pi$
$\xi_{31} = -4/\pi$	$\xi_{32} = 0$	$\xi_{41} = -2/\pi$	$\xi_{42} = 0$
$\xi_{51} = 0$	$\xi_{52} = 4/\pi$	$\xi_{61} = 2/\pi$	$\xi_{62} = 4/\pi$
$\xi_{71} = -2/\pi$	$\xi_{72} = 0$	$\xi_{81} = 2/\pi$	$\xi_{82} = 4/\pi$
$\xi_{91} = 0$	$\xi_{92} = 4/\pi$		

we obtain the following P_i and F_i for $i = 1, ..., 9$:

$$P_1 = \begin{bmatrix} 0.0044 & 0.0005 & -0.0002 & -0.0000 \\ 0.0005 & 0.0001 & 0.0000 & 0.0000 \\ -0.0002 & 0.0000 & 0.0180 & 0.0011 \\ -0.0000 & 0.0000 & 0.0011 & 0.0001 \end{bmatrix} > 0,$$

$$P_2 = \begin{bmatrix} 0.0063 & 0.0007 & -0.0003 & -0.0000 \\ 0.0007 & 0.0001 & 0.0000 & 0.0000 \\ -0.0003 & 0.0000 & 0.0238 & 0.0014 \\ -0.0000 & 0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_3 = \begin{bmatrix} 0.0052 & 0.0006 & -0.0003 & -0.0000 \\ 0.0006 & 0.0001 & 0.0000 & 0.0000 \\ -0.0003 & 0.0000 & 0.0206 & 0.0012 \\ -0.0000 & 0.0000 & 0.0012 & 0.0001 \end{bmatrix} > 0,$$

$$P_4 = \begin{bmatrix} 0.0081 & 0.0008 & -0.0004 & -0.0001 \\ 0.0008 & 0.0001 & 0.0000 & -0.0000 \\ -0.0004 & 0.0000 & 0.0286 & 0.0014 \\ -0.0001 & -0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_5 = \begin{bmatrix} 0.0066 & 0.0007 & -0.0004 & -0.0000 \\ 0.0007 & 0.0001 & 0.0001 & 0.0000 \\ -0.0004 & 0.0001 & 0.0249 & 0.0014 \\ -0.0000 & 0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_6 = \begin{bmatrix} 0.0073 & 0.0007 & -0.0004 & -0.0001 \\ 0.0007 & 0.0001 & 0.0001 & -0.0000 \\ -0.0004 & 0.0001 & 0.0268 & 0.0015 \\ -0.0001 & -0.0000 & 0.0015 & 0.0001 \end{bmatrix} > 0,$$

$$P_7 = \begin{bmatrix} 0.0098 & 0.0008 & -0.0005 & -0.0002 \\ 0.0008 & 0.0001 & -0.0001 & -0.0000 \\ -0.0005 & -0.0001 & 0.0321 & 0.0018 \\ -0.0002 & -0.0000 & 0.0018 & 0.0001 \end{bmatrix} > 0,$$

$$P_8 = \begin{bmatrix} 0.0089 & 0.0007 & -0.0003 & -0.0001 \\ 0.0007 & 0.0001 & -0.0001 & -0.0000 \\ -0.0003 & -0.0001 & 0.0303 & 0.0016 \\ -0.0001 & -0.0000 & 0.0016 & 0.0001 \end{bmatrix} > 0,$$

$$P_9 = \begin{bmatrix} 0.0095 & 0.0006 & -0.0004 & 0.0001 \\ -0.0006 & 0.0001 & 0.0001 & 0.0000 \\ -0.0004 & 0.0001 & 0.0316 & 0.0002 \\ 0.0001 & 0.0000 & 0.0002 & 0.0001 \end{bmatrix} > 0$$

$$F_1 = 10^4 \begin{bmatrix} 0.1972 & 0.0304 & 1.5274 & 0.0935 \\ -0.1281 & -0.0118 & 1.0172 & 0.0685 \end{bmatrix},$$

$$F_2 = 10^3 \begin{bmatrix} 7.1742 & 0.7969 & -0.8884 & -0.0548 \\ 0.9590 & 0.1227 & 7.8326 & 0.5126 \end{bmatrix},$$

$$F_3 = 10^3 \begin{bmatrix} 6.0170 & 0.6897 & -3.3359 & -0.2237 \\ -2.7700 & -0.3101 & 3.3993 & 0.2093 \end{bmatrix},$$

$$F_4 = 10^3 \begin{bmatrix} 8.4037 & 0.8734 & 0.3998 & -0.0322 \\ 0.2103 & 0.0442 & 7.2717 & 0.3699 \end{bmatrix},$$

$$F_5 = 10^3 \begin{bmatrix} 9.4557 & 1.0430 & 1.7476 & 0.0815 \\ 5.8559 & 0.6682 & 5.5596 & 0.3062 \end{bmatrix},$$

$$F_7 = 10^3 \begin{bmatrix} 5.1527 & 0.4600 & 1.4825 & -0.0170 \\ -3.5642 & -0.3247 & 2.7930 & 0.2575 \end{bmatrix},$$

$$F_6 = 10^3 \begin{bmatrix} 6.9498 & 0.7022 & 1.4199 & 0.0196 \\ 0.8046 & 0.1075 & 6.1771 & 0.3639 \end{bmatrix},$$

$$F_8 = 10^3 \begin{bmatrix} -3.5642 & -0.3247 & 2.7930 & 0.2575 \\ -0.0917 & -0.0166 & 4.9180 & 0.2621 \end{bmatrix},$$

$$F_9 = 10^3 \begin{bmatrix} 4.7285 & -0.1862 & 1.7978 & 0.2830 \\ 0.4183 & 0.2383 & -5.1097 & 0.5124 \end{bmatrix}$$

Satisfactory and less conservative results are obtained, showing the effectiveness of our proposal. Figures 4.4 and 4.5 show, respectively, the two links dynamic performances and the control torques, for the initial values $x(0) = \begin{bmatrix} \pi/3 & 0 & \pi/6 & 0 \end{bmatrix}^T$.

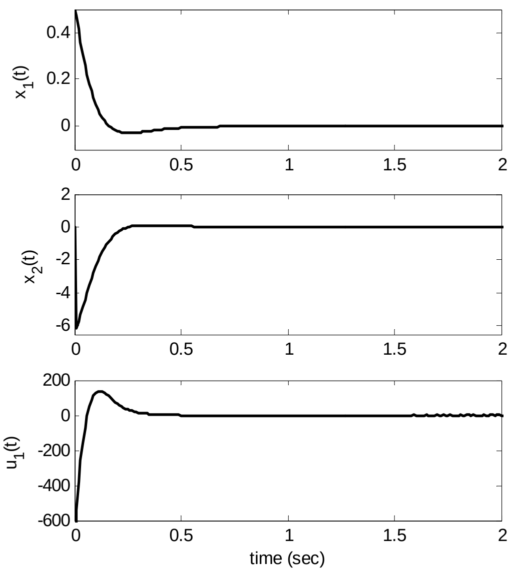

Figure 4.4: Example 4 (Two-link robot) performances: Link 1 (non-quadratic approach)

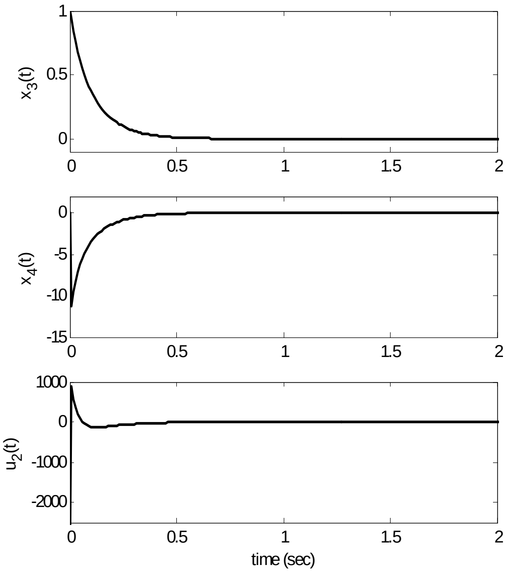

Figure 4.5: Example 4 (Two-link robot) performances: Link 2 (non-quadratic approach)

To better show the conservatism reduction of the non-quadratic approach, we attempt to stabilize the same system using the quadratic approach. The following simulation results are obtained:

$$P = \begin{bmatrix} 0.0008 & 0.0004 & -0.0004 & -0.0001 \\ 0.0004 & 0.0003 & -0.0002 & -0.0001 \\ -0.0004 & -0.0002 & 0.0104 & 0.0031 \\ -0.0001 & -0.0001 & 0.0301 & 0.0025 \end{bmatrix} > 0$$

$$F_1 = 10^8 \begin{bmatrix} 0.0010 & -1.0149 & 0.0001 & 2.0290 \\ 0.0002 & -1.0141 & -0.0000 & 1.0150 \end{bmatrix},$$

$$F_2 = 10^8 \begin{bmatrix} 0.0009 & -0.0005 & 0.0000 & 1.0141 \\ 0.0002 & -0.5074 & -0.0000 & 0.0003 \end{bmatrix},$$

$$F_3 = 10^8 \begin{bmatrix} 0.0009 & 1.0143 & 0.0000 & 2.0283 \\ -0.0001 & -1.0149 & 0.0000 & -1.0141 \end{bmatrix},$$

$$F_4 = 10^8 \begin{bmatrix} 0.0011 & -0.0008 & 0.0000 & 1.0143 \\ -0.0001 & -0.5071 & 0.0000 & 0.0003 \end{bmatrix},$$

$$F_5 = 10^8 \begin{bmatrix} 0.0011 & -1.0152 & 0.0000 & 2.0286 \\ 0.0001 & -1.0140 & 0.0000 & 1.0145 \end{bmatrix},$$

$$F_6 = 10^8 \begin{bmatrix} 0.0011 & -0.0009 & 0.0000 & 1.0143 \\ -0.0001 & -0.5071 & -0.0000 & 0.0003 \end{bmatrix},$$

$$F_7 = 10^8 \begin{bmatrix} 0.0009 & 1.0140 & 0.0000 & 2.0285 \\ -0.0002 & -1.0146 & -0.0001 & -1.0140 \end{bmatrix},$$

$$F_8 = 10^8 \begin{bmatrix} 0.0009 & -0.0004 & 0.0000 & 1.0141 \\ 0.0001 & -0.5074 & 0.0000 & 0.0002 \end{bmatrix},$$

$$F_9 = 10^8 \begin{bmatrix} 0.0009 & -1.0143 & -0.0000 & 2.0288 \\ 0.0001 & -1.0136 & -0.0001 & 1.0148 \end{bmatrix}$$

Figures 4.6 and 4.7 show the failure of the quadratic approach to stabilize the two-links robot, whereas, figures 4.8 and 4.9 show $\dot{V}(x(t))$ for respectively the fuzzy Lyapunov function and the quadratic Lyapunov function.

$$\text{At } t = 2sec,\ \dot{V}(x(t)) \begin{cases} -1.1054 * 10^{-14} < 0; \text{ for the non-quadratic approach} \\ 0.1138 * 10^{5} > 0; \text{ for the quadratic approach} \end{cases}$$

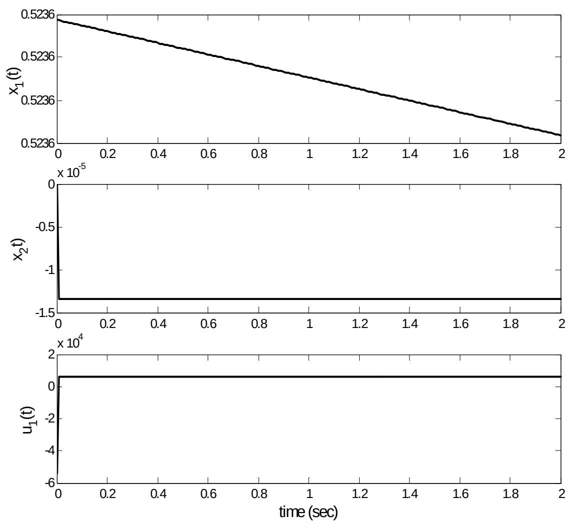

Figure 4.6: Example 4 (Two-links robot) performances: Link 1 (quadratic approach)

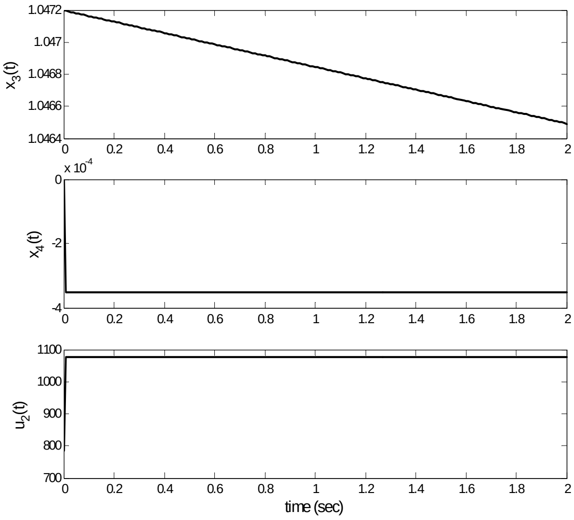

Figure 4.7: Example 4 (Two-links robot) performances: Link 2 (quadratic approach)

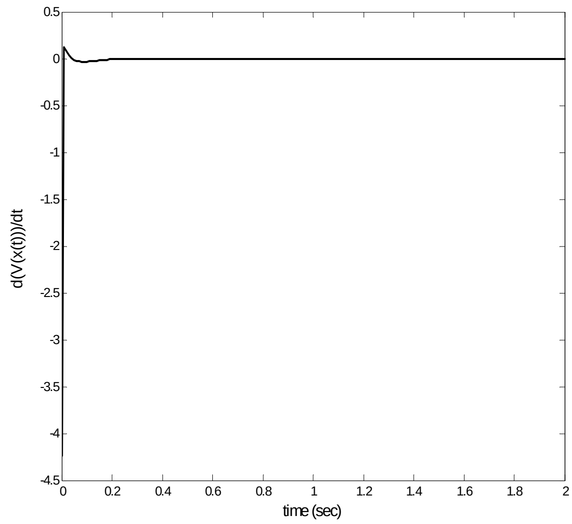

Figure 4.8: $\dot{V}\left(x\left(t\right)\right)$ with non-quadratic approach

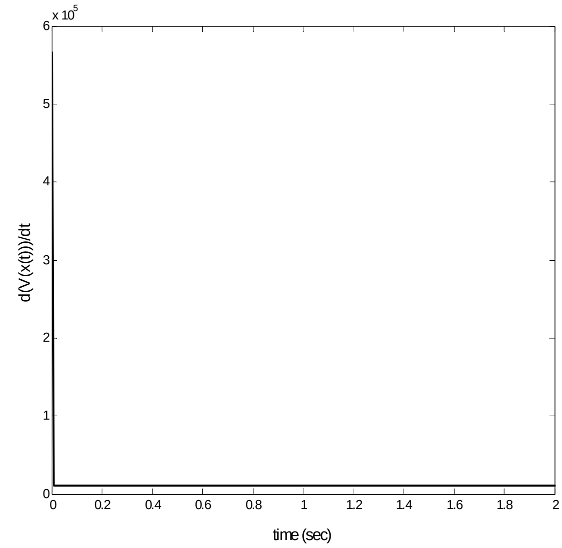

Figure 4.9: $\dot{V}(x(t))$ with quadratic stabilization

4.5 Conclusion

This chapter has presented a new fuzzy Lyapunov approach for the stabilization of T-S fuzzy systems, basing on the fuzzy Lyapunov function, which is defined by fuzzily blending quadratic Lyapunov functions. The conditions are derived in a logical way while profiting fully of the fuzzy Lyapunov function advantage and considering two assumptions that are a proportional relation between the multiple quadratic Lyapunov functions and an upper bound for the time derivative of the premise membership function whose corresponding LMIs that support it in theorem (14) are solved with those of theorem (13) to stabilize the closed loop fuzzy system. Hence, the PDC local feedback gains construction procedure is simple and can be solved effectively by optimization computation tools. The proposed approach leads to less conservative results and very good results are obtained for different examples, even for those that do not admit a single Lyapunov function and for complicated systems such as a two-links robot with 9 rules, thus illustrating the effectiveness of the proposed stabilization approach.

Chapter 5

Output Stabilization of Takagi-Sugeno Fuzzy Systems via Fuzzy Observer

5.1 Introduction

The states of a system are not always available for measurement which is the case in a lot of practical problems. To overcome this limit, the notion of observer was introduced. The concept of linear regulator and linear observer were introduced by Kalman (Kalman, 1961) for linear systems in stochastic environment and by Luenberger (Luenberger, 1966) for deterministic linear systems such that the difference between the system state $x(t)$ and the observer state $\hat{x}(t)$ converges to zero when t tends to ∞. For nonlinear systems, different observer designs were proposed such as the extended kalman observer, the sliding mode observer (Utkin & Drakunov, 1995), the high gain observer (Nicosia & Tornambe, 1989) and the T-S fuzzy observer, that was introduced by several authors in the literature such as Tanaka (Tanaka & Sano, 1994), Feng et al. (Feng *et al.* , 1997) and Jadbabaie (Jadbabaie, 1997, b) who proposed fuzzy observers with an asymptotic convergence. Tanaka proposed in his paper (Tanaka *et al.* , 1998) a globally exponentially stable fuzzy controllers and fuzzy observers designs for continuous and discrete fuzzy systems for both measurable and non measurable premise variables. Other approaches were proposed by different authors (Fayaz, 2000) (Ma & Sun, 2001) (Cao & Frank, 2000),(Chen & Liu, 2004),(Wang, 2004),(Chen & Liu, 2005),(Lin *et al.* , 2006) and (Lin *et al.* , 2008) with other considerations. Hence, the observer design is a very important problem in control systems and the stability of the whole system, with the fuzzy controller and the fuzzy observer, must be guaranteed. For a T-S fuzzy system, a separation property is used to check the stability of the global system. This concept was introduced by Jadbabaie et al. (Jadbabaie, 1997, b) and Ma et al. (Ma *et al.* , 1998) by two different approaches to assure an independent design for the controller and the observer while assuring the stability of the global T-S fuzzy system.

In this chapter, we extend the stability results given in (Abdelmalek *et al.* , 2007) to the case when the states are not available for measurement and feedback in other terms fuzzy observer, by guarantying the stability of the whole system (Abdelmalek & Goléa, 2009). The observer design is based on fuzzy implications, with fuzzy sets in antecedents, and a Lunenberger observer form in the consequents. Each fuzzy rule is responsible for observing the states of a locally linear sub-system (Jadbabaie, 1997, a),(Jadbabaie, 1997, b), then a separation property is used to check the stability of the global system. We applied in our proposal the separation principle of Ma et al.(Ma *et al.* , 1998), due to its simplicity, since it does not depend on the stability conditions but rather on the fuzzy Lyapunov functions. Indeed, the separation principle design proposed in (Jadbabaie, 1997, b) is not appropriated for the case of different stability conditions.

5.2 Fuzzy observer design

An observer is required to satisfy $x(t)-\hat{x}(t) \rightarrow 0$ when $t \rightarrow \infty$, where $\hat{x}(t)$ denotes the states vector estimated by a Luenberger observer for a linear time invariant system, given by:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu + K_i (y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}$$

A fuzzy observer is designed by fuzzy IF-THEN rules whose consequents are of Luenberger observer form. Thus, the *ith* observer rule is of the following form:

$$\begin{array}{c} \text{Rule } i : \text{IF } z_1(t) \text{ is } M_{i1} \text{ and ... and } z_p(t) \text{ is } M_{ip} \\ \text{THEN} \left\{ \begin{array}{c} \dot{\hat{x}}(t) = A_i\hat{x}(t) + B_i u(t) + K_i (y(t) - \hat{y}(t)) \\ \hat{y}_i(t) = C_i \hat{x}(t) \end{array} \right. \quad i = 1, 2, ..., r \end{array} \tag{5.1}$$

where $z(t) = [z_1(t), ..., z_p(t)]$ is the premise variable, K_i $i = 1, ..., r$ are the observation error matrices. $y(t)$ and $\hat{y}(t)$ are the final outputs of the fuzzy system and the fuzzy observer, respectively. The final outputs of the fuzzy observer, that are obtained by fuzzy blending of the local observer of Luenberger form, depend on the dependence of the premise variables on the state variables, hence two cases can be considered:

- Case 1: $z_1(t), ..., z_p(t)$ are measurable, so they do not depend on the estimated variables, then the normalized weights of the observer $h_i(\hat{z}(t))$ are replaced by $h_i(z(t))$ (Tanaka *et al.* , 1998),(Ma *et al.* , 1998).

- Case 2: $z_1(t), ..., z_p(t)$ are not measurable, then they depend on the estimated variables. They must be estimated to allow the calculation of normalized weights of the observer $h_i(\hat{z}(t))$ that are the same normalized weight of the fuzzy model for each rule with $\hat{z}_1, \hat{z}_2, ..., \hat{z}_p$, representing the estimated premise variables.

Hence, there is a difference between the designs of the two cases when using $z(t)$ or $\hat{z}(t)$.

5.2.1 Case 1: measurable premise variables

The fuzzy observer is inferred as follows:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i\hat{x}(t) + B_iu(t) + K_i(y(t) - \hat{y}(t))\right) \tag{5.2}$$

$$\hat{y}(t) = \sum_{i=1}^{r} h_i(z(t))\,C_i\hat{x}(t) \tag{5.3}$$

We use the same weight $h_i(z(t))$ of the ith rule of the fuzzy model (3.20) and (3.21). The fuzzy observer design is to find the local gains K_i in the consequent part. The PDC fuzzy controller takes the following form:

$$u(t) = -\frac{\sum_{i=1}^{r} w_i(z(t))\,F_i\hat{x}(t)}{\sum_{i=1}^{r} w_i(z(t))} = -\sum_{i=1}^{r} h_i(z(t))\,F_i\hat{x}(t) \tag{5.4}$$

Replacing this fuzzy controller in the fuzzy observer expression (3.20) and considering $\tilde{x}(t) = x(t) - \hat{x}(t)$, we obtain the following representation:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\left\{(A_i - B_iF_j)\,x(t) + B_iF_j\tilde{x}(t)\right\}, \tag{5.5}$$

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\left\{A_i - K_iC_j\right\}\tilde{x}(t) \tag{5.6}$$

The augmented system representations are given by:

$$\begin{aligned}
\dot{x_a}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, G_{ij} x_a(t) \\
&= \sum_{i=1}^{r} h_i(z(t))\, h_i(z(t))\, G_{ii} x_a(t) \\
&\quad +2\sum_{i=1}^{r}\sum_{i<j} h_i(z(t))\, h_j(z(t))\, \frac{G_{ij}+G_{ji}}{2} x_a(t)
\end{aligned} \tag{5.7}$$

where $x_a = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$ and $G_{ij} = \begin{bmatrix} A_i - B_i F_j & B_i F_j \\ 0 & A_i - K_i C_j \end{bmatrix}$

Tanaka et al. (Tanaka *et al.* , 1998) proposed the following theorem for the global asymptotic stability of the augmented system.

Theorem 15 *The equilibrium of the augmented system described by (5.7) is globally asymptotically stable if there exists a common positive definite matrix P such that*

$$G_{ii}^T P + P G_{ii} < 0, \tag{5.8}$$

$$\frac{(G_{ij}+G_{ji})^T}{2} P + P \frac{(G_{ij}+G_{ji})}{2} < 0, \tag{5.9}$$

$$\forall\ i < j \ s.t.\ h_i \cap h_j \neq \varnothing$$

5.2.2 Case 2: non measurable premise variables

The fuzzy observer is inferred as follows:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(\hat{z}(t))\left(A_i \hat{x}(t) + B_i u(t) + K_i\left(y(t) - \hat{y}(t)\right)\right) \tag{5.10}$$

$$\hat{y}(t) = \sum_{i=1}^{r} h_i(\hat{z}(t))\, C_i \hat{x}(t) \tag{5.11}$$

The PDC fuzzy controller takes the following form:

$$u(t) = -\frac{\sum_{i=1}^{r} w_i(\hat{z}(t))\, F_i \hat{x}(t)}{\sum_{i=1}^{r} w_i(\hat{z}(t))} = -\sum_{i=1}^{r} h_i(\hat{z}(t))\, F_i \hat{x}(t) \tag{5.12}$$

The augmented system is then given by:

$$\begin{aligned}\dot{x_a}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{r} h_i(z(t))\,h_j(\hat{z}(t))\,h_s(\hat{z}(t))\,G_{ijs}x_a(t)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(\hat{z}(t))\,h_j(\hat{z}(t))\,G_{ijj}x_a(t)\\ &\quad +2\sum_{i=1}^{r}\sum_{j<s} h_i(z(t))\,h_j(\hat{z}(t))\,h_s(\hat{z}(t))\,\frac{G_{ijs}+G_{isj}}{2}x_a(t)\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}x_a &= \begin{bmatrix} x(t)\\ \tilde{x}(t)\end{bmatrix}\\ \tilde{x}(t) &= x(t)-\hat{x}(t)\\ G_{ijs} &= \begin{bmatrix} A_i-B_iF_s & B_iF_s\\ S^1_{ijs} & S^2_{ijs}\end{bmatrix}\\ S^1_{ijs} &= (A_i-A_j)-(B_i-B_j)F_s+K_j(C_s-C_i)\\ S^2_{ijs} &= A_j-K_jC_s+(B_i-B_j)F_s\end{aligned}$$

Stability conditions for the augmented system (5.13) are given by Tanaka et al (Tanaka *et al.* , 1998):

Theorem 16 *The equilibrium of the augmented system described by (5.13) is globally asymptotically stable if there exists a common positive definite matrix P such that*

$$G_{ijj}^T P + PG_{ijj} < 0, \tag{5.14}$$

$$\frac{(G_{ijs}+G_{isj})^T}{2}P + P\frac{(G_{ijs}+G_{isj})}{2} < 0, \quad j<s \tag{5.15}$$

$$\forall\ i,\ j<j\ s.t.\ h_i\cap h_j\cap h_s \neq \varnothing \tag{5.16}$$

It is interesting to note that the control gains and the observer gains can be designed separately and then using a separation property check the stability of the whole system. However, this principle is applicable only in case A, i.e. $z(t)=\hat{z}(t)$.

5.3 Proposed continuous Fuzzy observer design

Our proposal is based on continuous T-S fuzzy models and on the first case where $z(t)=\hat{z}(t)$ (Abdelmalek & Goléa, 2009). The controller is based on the estimated state and is given by (5.4), then the closed loop system is given by:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\,(A_i x(t) - B_i F_j \hat{x}(t)) \tag{5.17}$$

On the other hand, by substituting $y(t) = \sum_{i=1}^{r} h_i(z(t))\, C_i x(t)$ and (5.3) in (5.2) we obtain:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} h_i(z(t))\,(A_i\hat{x}(t) + B_i u(t)) + \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, K_i C_i\,(x(t) - \hat{x}(t)) \tag{5.18}$$

that can be written as

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\,[(A_i - K_i C_j)\,\hat{x}(t) + B_i u(t) + K_i C_j x(t)] \tag{5.19}$$

Defining the steady error as $\tilde{x} = x - \hat{x}$, and substracting (5.17) from (5.19), we obtain:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\,(A_i - K_i C_j)\,\tilde{x}(t) \tag{5.20}$$

The design of the fuzzy observer is to determine the local gains K_i, using stability conditions of theorem (17) such that the steady error tends to zero.

Theorem 17 *The observer dynamic is stable if there exist positive definite matrices* $Po_1, Po_2, \ldots, Po_r$ *and matrices* $K_1, K_2, \ldots, K_r$ *such that the following is satisfied:*

$$Po_i > 0, \qquad i = 1, 2, .., r \tag{5.21}$$

$$\sum_{\rho=1}^{r} \phi_\rho Po_\rho + \left(G_{jj}^T Po_i + Po_i G_{jj}\right) < 0, \qquad i, j = 1, 2, .., r \tag{5.22}$$

$$\left\{\frac{G_{jk} + G_{kj}}{2}\right\}^T Po_i + Po_i \left\{\frac{G_{jk} + G_{kj}}{2}\right\} < 0, \quad \forall i, j, k \in \{1, 2, .., r\} \ s.t.\ j < k \tag{5.23}$$

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & Po_i^{-1} \end{bmatrix} \geq 0, \ for\ i = 1, .., r \tag{5.24}$$

$$\begin{bmatrix} \phi_\rho Po_i & W_{ij\rho\ell}^T \\ W_{ij\rho\ell} & \phi_\rho I \end{bmatrix} \geq 0, \ \forall i, j, \rho \in \{1, 2, .., r\} \ \forall \ell \tag{5.25}$$

where $G_{jk} = A_j - K_j C_k$, $G_{jj} = A_j - K_j C_j$ *and* $W_{ij\rho\ell} = \xi_{\rho\ell}(A_i - K_i C_j)$.

These inequalities can be recast in terms of LMIs by the following changes of variables:

$$\begin{cases} Po_i = Xo_i^{-1}, \ \forall i \in \{1,2,...,r\} \\ Xo_i = \alpha_{ij} Xo_j \text{ such that } \alpha_{ij} = 1/\alpha_{ji}, \forall i,j \in \{1,2,...,r\} \ and \ i \neq j \\ K_i = \beta_{ij} K_j \text{ such that } \beta_{ij} = 1/\beta_{ji}, \forall i,j \in \{1,2,...,r\} \ and \ i \neq j \\ Ni = K_i C_i Xo_i, \forall i \in \{1,2,...,r\} \end{cases}$$

The coefficients α_{ij}, β_{ij} and ϕ_ρ for $i, j, \rho = 1, 2, ..., r$ and $i \neq j$, can be chosen heuristically according to the considered application. α_{ij} and β_{ij} must be different from 1 (for $i = j$; $\alpha_{ij} = \beta_{ij} = 1$) and selection of $\xi_{\rho\ell}$ is obtained from $\dot{h}_i(z(t))$. Subsequently, we will consider that the premise variables do not depend on the estimated states $\hat{x}(t)$.

5.4 Separation property of observer/controller

By augmenting the states of the system with the state estimation error, we obtain the following $2n$ dimensional state equations for the observer/controller closed-loop system:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} &= \begin{bmatrix} \sum_{i=1}^r \sum_{j=1}^r h_i h_j (A_i - B_i F_j) & \sum_{i=1}^r \sum_{j=1}^r h_i h_j B_i F_j \\ 0 & \sum_{i=1}^r \sum_{j=1}^r h_i h_j (A_i - K_i C_j) \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \\ y &= \begin{bmatrix} \sum_{i=1}^r h_i C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \end{aligned} \tag{5.26}$$

To show that the whole system above is stable, we must show that the separation property holds. For this purpose we suggest to extend the separation property principle proposed by Ma et al. in their paper (Ma *et al.* , 1998) to the non-quadratic design that we propose in (Abdelmalek *et al.* , 2007). We have to construct a comparison system $\dot{w} = Aw$, where A is a function of γ_i and $\widetilde{\gamma}_i$, $i = 1, 2, 3, 4$. Then using the vector comparison principle, we can obtain a global system globally asymptotically stable. The separation property is expressed by the following theorem:

Theorem 18 *(Ma* et al. *, 1998): If there exist two scalar functions $V(x) : R^n \to R$ and $\widetilde{V}\left(\widetilde{x}\right) : R^n \to R$ and positive real numbers $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ and $\widetilde{\gamma}_4$ such that*

$$\gamma_1 \|x\|^2 \leq V(x) \leq \gamma_2 \|x\|^2, \ \widetilde{\gamma}_1 \left\|\widetilde{x}\right\|^2 \leq \widetilde{V}\left(\widetilde{x}\right) \leq \widetilde{\gamma}_2 \left\|\widetilde{x}\right\|^2 \tag{5.27}$$

$$\begin{aligned} \frac{\partial V(x)}{\partial x} \sum_{i=1}^r \sum_{j=1}^r h_i h_j (A_i - B_i F_j) x &\leq -\gamma_3 \|x\|^2, \\ \frac{\partial \widetilde{V}\left(\widetilde{x}\right)}{\partial \widetilde{x}} \sum_{i=1}^r \sum_{j=1}^r h_i h_j (A_i - K_i C_j) \widetilde{x} &\leq -\widetilde{\gamma}_3 \left\|\widetilde{x}\right\|^2 \end{aligned} \tag{5.28}$$

$$\left\| \frac{\partial V(x)}{\partial x} \right\| \leq \gamma_4 \|x\|, \quad \left\| \frac{\partial \tilde{V}\left(\tilde{x}\right)}{\partial \tilde{x}} \right\| \leq \tilde{\gamma}_4 \left\| \tilde{x} \right\| \tag{5.29}$$

then, the whole system is globally asymptotically stable.

Hence, this theorem shows that the fuzzy controller and the fuzzy observer can be designed to be stable independently and the whole system that is fuzzy controller, fuzzy observer and fuzzy system is still stable. This theorem is extended to the case of non-quadratic stability conditions, where $V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\, x^T(t)\, P_i x(t)$ and $\tilde{V}\left(\tilde{x}\right) = \sum_{i=1}^{r} h_i(z(t))\, \tilde{x}^T(t)\, Po_i \tilde{x}(t)$ (Abdelmalek & Goléa, 2009). The principle of this method is to find the scalars $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$and $\tilde{\gamma}_4$ that satisfy inequalities (5.27)-(5.29) and then to satisfy the following inequality:

$$\begin{bmatrix} \dot{V}(x(t)) \\ \dot{\tilde{V}}\left(\tilde{x}\right) \end{bmatrix} \leq \begin{bmatrix} -\frac{\gamma_3}{2\gamma_2} & \frac{a\gamma_4^2}{2\gamma_3\tilde{\gamma}_1} \\ 0 & -\frac{\tilde{\gamma}_3}{\tilde{\gamma}_2} \end{bmatrix} \begin{bmatrix} V(x(t)) \\ \tilde{V}\left(\tilde{x}\right) \end{bmatrix} = A \begin{bmatrix} V(x(t)) \\ \tilde{V}\left(\tilde{x}\right) \end{bmatrix} \tag{5.30}$$

where

$$A = \begin{bmatrix} -\frac{\gamma_3}{2\gamma_2} & \frac{a\gamma_4^2}{2\gamma_3\tilde{\gamma}_1} \\ 0 & -\frac{\tilde{\gamma}_3}{\tilde{\gamma}_2} \end{bmatrix} \tag{5.31}$$

is a stability matrix. Hence the construction of the comparison system $\dot{w} = Aw$, which is obviously globally asymptotically stable and the use of the vector comparison principle, allow us to verify that the whole system (5.26) is globally asymptotically stable (the proof of theorem 18 is given in (Ma *et al.* , 1998)).

5.5 Design examples

This part presents the design examples that illustrates the effectiveness of the proposed controller-observer design with the separation property principle for the stability checking of the global system, i.e. fuzzy system, fuzzy controller and fuzzy observer (Abdelmalek & Goléa, 2009).

5.5.1 Example 1: *The inverted Pendulum*

We recall that this system (3.33) is modeled by the following two fuzzy rules:

Rule 1 : IF $x_1(t)$ is about 0 THEN $\dot{x}(t) = A_1 x(t) + B_1 u(t)$, $y(t) = C_1 x(t)$

Rule 2 : IF $x_1(t)$ is about $\pm\pi/2$ $(|x_1| < \pi/2)$ THEN$\dot{x}(t) = A_2 x(t) + B_2 u(t)$, $y(t) = C_2 x(t)$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{2g}{4l/3-aml} & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi\left(4l/3-aml\beta^2\right)} & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{bmatrix}, \ \beta = \cos\left(88^\circ\right), \\ C_1 &= C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

The control objective for this example is to balance the inverted pendulum for the approximate range $x_1 \in (-\pi/2, \pi/2)$. The PDC control laws are as follows:

$$\begin{aligned} &\text{Rule 1 : IF } x_1(t) \text{ is about 0 THEN } u(t) = -F_1\hat{x}(t) \\ &\text{Rule 2 : IF } x_1(t) \text{ is about } \pm\pi/2 \ \ (|x_1| < \pi/2) \text{ THEN } u(t) = -F_2\hat{x}(t) \end{aligned}$$

The observer rules are:

$$\begin{aligned} &\text{Rule 1 : IF } x_1(t) \text{ is about 0 THEN } \dot{\hat{x}}(t) = A_1\hat{x}(t) + B_1u(t) + K_1C_1\left(x(t) - \hat{x}(t)\right) \\ &\text{Rule 2 : IF } x_1(t) \text{ is about } \pm\pi/2 \ \ (|x_1| < \pi/2) \text{ THEN } \begin{array}{c} \dot{\hat{x}}(t) = A_2\hat{x}(t) + B_2u(t) \\ +K_2C_2\left(x(t) - \hat{x}(t)\right) \end{array} \end{aligned}$$

Hence the control law that guarantees the stability of the fuzzy model and the fuzzy observer system is given by:

$$u(t) = -h_1\left(x_1(t)\right)F_1\hat{x}(t) - h_2\left(x_1(t)\right)F_2\hat{x}(t) \tag{5.32}$$

where $h_1$and h_2 are the membership functions of triangular form, for rules 1 and 2 respectively.

Applying the proposed approach, the objective of balancing and stabilizing the pendulum and the estimation process are reached with success for different initial conditions of $x_1(0) \in (-\pi/2, \pi/2)$ and $x_2(0) = 0$. We considered two cases:

a) with a pole placement:

We choose the closed-loop eigenvalues $\begin{bmatrix} -3.0 & -5.0 \end{bmatrix}$ for $(A_1 - B_1F_1)$ and $(A_2 - B_2F_2)$ and the closed-loop eigenvalues $\begin{bmatrix} -50.0 & -60.0 \end{bmatrix}$ for $(A_1 - K_1C_1)$ and $(A_2 - K_2C_2)$,we have then:

$$F_1 = \begin{bmatrix} -645.8824 & -160.0000 \end{bmatrix}, F_2 = 10^3 \begin{bmatrix} -4.6525 & -1.5279 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 110.0000 & 174.4694 \end{bmatrix}^T, K_2 = \begin{bmatrix} 110.0000 & 321.5121 \end{bmatrix}^T$$

Hence, for

$$\begin{aligned}\phi_1 &= \phi_2 = 0.5 \\ \xi_{11} &= -0.0637,\ \xi_{12} = 0.0637,\ \xi_{21} = -0.0637,\ \xi_{22} = 0.0637\end{aligned}$$

and for the initial condition $x(0) = [\pi/6, 0]^T$, we obtain the following results:

For the controller design:

$$P_1 = 10^8 \begin{bmatrix} 3.0097 & 0.1270 \\ 0.1270 & 0.1262 \end{bmatrix} > 0,\ P_2 = 10^8 \begin{bmatrix} 2.9974 & 0.1267 \\ 0.1267 & 0.1259 \end{bmatrix} > 0$$

For the observer design:

$$Po_1 = 10^7 \begin{bmatrix} 7.3509 & -1.0523 \\ -1.0523 & 0.5042 \end{bmatrix} > 0,\ Po_2 = 10^7 \begin{bmatrix} 7.3102 & -1.0432 \\ -1.0432 & 0.5000 \end{bmatrix} > 0$$

Figures (5.1) and (5.2) show the closed loop behavior of the fuzzy controller and the fuzzy observer, for respectively, the inverted pendulum position, velocity and control force evolution of the closed loop system.

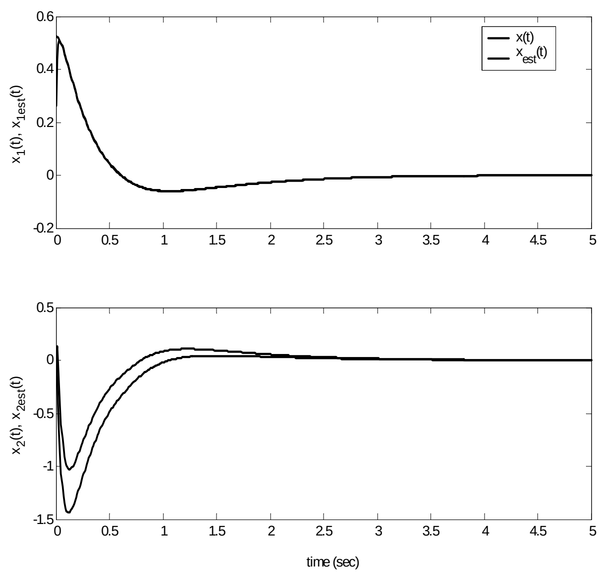

Figure 5.1: Inverted pendulum performances with a pole placement

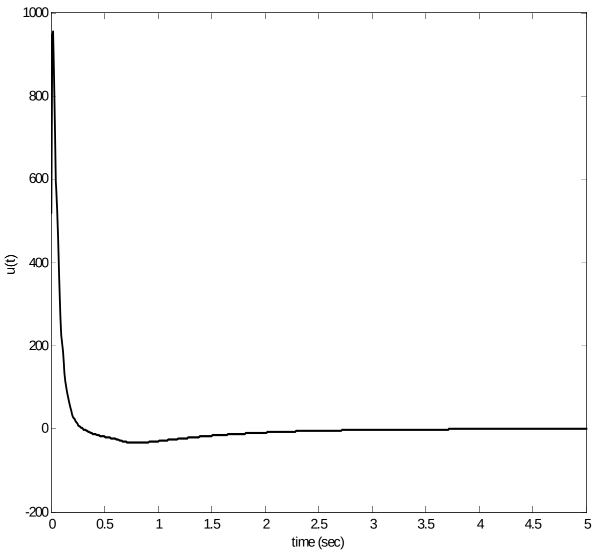

Figure 5.2: Inverted pendulum action evolution with a pole placement

The stability of the whole system (fuzzy controller+fuzzy observer+fuzzy system) is verified applying the vector comparison principle, having two scalar functions $V(x)$, $\widetilde{V}\left(\widetilde{x}\right)$ and positive real numbers obtained from simulation and that satisfy the inequalities (5.27)-(5.29), their values are:

$\gamma_1 = 4.6738 \times 10^7$	$\gamma_2 = 4.6039 \times 10^8$	$\gamma_3 = 1.2411 \times 10^8$	$\gamma_4 = 4.3764 \times 10^8$
$\widetilde{\gamma}_1 = 1.3174 \times 10^4$	$\widetilde{\gamma}_2 = 4.4097 \times 10^8$	$\widetilde{\gamma}_3 = 5.0038 \times 10^6$	$\widetilde{\gamma}_4 = 4.3820 \times 10^8$

Comparing our results with those obtained for the same example with a pole placement in (Jadbabaie, 1997, a), our results are very interesting, since on one side the stability design that depends on non-quadratic stability conditions is less conservative and on the other side the separation property design is very flexible since it do not depends on the stability conditions but directly on the fuzzy Lyapunov functions.

b) without a pole placement:

For

$$\begin{aligned} \phi_1 &= \phi_2 = 1 \\ \alpha_{12} &= 0.4,\ \alpha_{21} = 1/\alpha_{12} \\ \beta_{12} &= 1.2,\ \beta_{21} = 1/\beta_{12} \\ \xi_{11} &= -0.0064,\ \xi_{12} = 0.0064,\ \xi_{21} = -0.0064,\ \xi_{22} = 0.0064 \end{aligned}$$

we obtain the following $P_1, P_2, F_1, F_2, Po_1, Po_2, K_1$ and K_2 for the initial condition $x(0) =$

$\begin{bmatrix} \pi/3 & 0 \end{bmatrix}^T$:

$$P_1 = \begin{bmatrix} 0.6122 & 0.2216 \\ 0.2216 & 0.0878 \end{bmatrix} > 0,\ P_2 = \begin{bmatrix} 0.6131 & 0.2181 \\ 0.2181 & 0.0794 \end{bmatrix} > 0$$

$$F_1 = \begin{bmatrix} -937.3591 & -294.8718 \end{bmatrix},\ F_2 = 10^3 \begin{bmatrix} -6.0636 & -2.0828 \end{bmatrix}$$

$$Po_1 = \begin{bmatrix} 0.1459 & -0.0034 \\ -0.0034 & 0.0095 \end{bmatrix} > 0,\ Po_2 = \begin{bmatrix} 0.0425 & -0.0061 \\ -0.0061 & 0.0062 \end{bmatrix} > 0$$

$$K_1 = \begin{bmatrix} 9.6602 & 24.7890 \end{bmatrix}^T,\ K_2 = \begin{bmatrix} 5.9401 & 12.6046 \end{bmatrix}^T$$

Also very good results are obtained for the stability of the whole system which is checked applying the vector comparison principle, and the positive real numbers values are:

$\gamma_1 = 0.0868$	$\gamma_2 = 2.0100 \times 10^8$	$\gamma_3 = 0.1475$	$\gamma_4 = 2.0327 \times 10^8$
$\widetilde{\gamma}_1 = 1.6547 \times 10^{-6}$	$\widetilde{\gamma}_2 = 1.9823 \times 10^8$	$\widetilde{\gamma}_3 = 0.0053$	$\widetilde{\gamma}_4 = 1.9619 \times 10^8$

Figures 5.3 and 5.4 show the closed loop behavior of the whole system (fuzzy system+fuzzy controller+fuzzy observer), for respectively, the inverted pendulum position, velocity and control force evolution for the initial condition above.

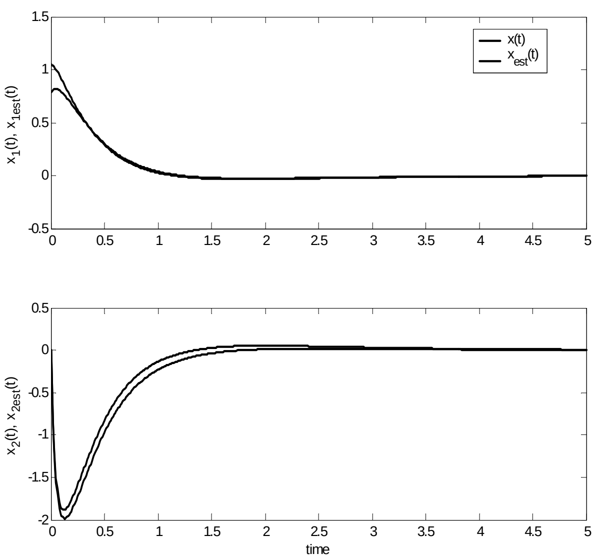

Figure 5.3: Inverted pendulum performances without pole placement

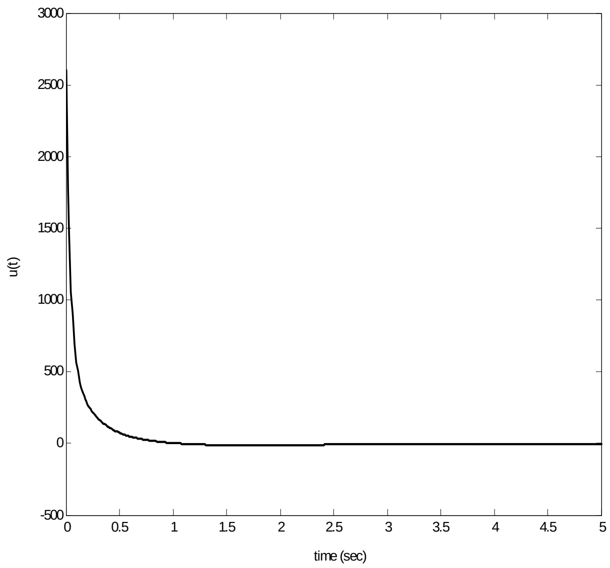

Figure 5.4: Inverted pendulum action evolution without pole placement

5.5.2 Example 2: *TORA system*

We choose to apply the proposed method to another example; the TORA system (Translational Oscillator with an eccentric Rotational proof mass actuator). The equations of motion are given by (Bupp *et al.* , 1998):

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= \frac{-x_1(t)+\epsilon x_4^2(t)\sin x_3(t)}{1-\epsilon^2\cos^2 x_3(t)} + \frac{-\epsilon\cos x_3(t)}{1-\epsilon^2\cos x_3(t)}u(t), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= \frac{\epsilon\cos x_3(t)\left(x_1(t)-\epsilon x_4^2(t)\sin x_3(t)\right)}{1-\epsilon^2\cos^2 x_3(t)} + \frac{1}{1-\epsilon^2\cos x_3(t)}u(t)
\end{aligned} \tag{5.33}$$

where $x_1(t)$ and $x_2(t)$ denote respectively the translational position and velocity of the cart, $x_3(t)$ and $x_4(t)$ denote respectively the angular position and velocity of the rotational proof mass. u is the torque applied to the eccentric mass and $\epsilon = 0.1$ in the numerical simulation. Hence, for $x = [x_1, x_2, x_3, x_4]^T$, we have the following T-S fuzzy model (Jadbabaie *et al.* , 1998, a):

Rule 1 : IF $x_1(t)$ is about 0 THEN $\dot{x}(t) = A_1x(t) + B_1u(t)$, $y(t) = C_1x(t)$

Rule 2 : IF $x_1(t)$ is about $\pm\pi/2$ THEN $\dot{x}(t) = A_2x(t) + B_2u(t)$, $y(t) = C_2x(t)$

where

$$A_1 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ -1/1-\epsilon^2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ \epsilon/1-\epsilon^2 & 0.0 & 0.0 & 0.0 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ -1/1-\epsilon^2\beta^2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ \epsilon*\beta/1-\epsilon^2\beta^2 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.0 \\ -\epsilon/1-\epsilon^2 \\ 0.0 \\ 1/1-\epsilon^2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0 \\ -\epsilon*\beta/1-\epsilon^2\beta^2 \\ 0.0 \\ 1/1-\epsilon^2\beta^2 \end{bmatrix}$$

$$C_1 = C_2 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix},$$

and $\beta = \cos(80°)$. The control law that guarantees the stability of the fuzzy model and the fuzzy observer system is given by (5.32), where $h_1$and h_2 are the membership functions of rules 1 and 2 of TORA system, respectively, they are given by:

$$\begin{aligned} h_1(x_1(t)) &= 1 - \frac{2}{\pi}|x_1(t)| \\ h_2(x_1(t)) &= \frac{2}{\pi}|x_1(t)|. \end{aligned}$$

The objective of stabilizing the fuzzy controller and the fuzzy observer is reached with success. First, two sets of LMIs are solved separately to obtain the gains F_i and K_i of respectively the fuzzy controller and the fuzzy observer with a guaranteed stability for each design. Then the global stability of the whole system is checked using the vector comparison principle and constructing the comparison system. Hence, for

$$\begin{aligned} \phi_1 &= \phi_2 = 0.1 \\ \alpha_{12} &= 0.5, \alpha_{21} = 1/\alpha_{12} \\ \beta_{12} &= 0.1, \beta_{21} = 1/\beta_{12} \\ \xi_{11} &= -0.0064, \xi_{12} = 0.0064, \xi_{21} = -0.0064, \xi_{22} = 0.0064 \end{aligned}$$

we obtain the following $P_1, P_2, F_1, F_2, Po_1, Po_2, K_1$ and K_2 for the initial condition $x(0) = \begin{bmatrix} 0.4 & 0 & \pi/3 & 0 \end{bmatrix}^T$:

$$P_1 = \begin{bmatrix} 6.3461 & -0.3446 & -0.0285 & -0.2009 \\ -0.3446 & 3.7592 & 0.0118 & 0.0567 \\ -0.0285 & 0.0118 & 0.0006 & 0.0022 \\ -0.2009 & 0.0567 & 0.0022 & 0.0146 \end{bmatrix} > 0,$$

$$P_2 = \begin{bmatrix} 6.3471 & -0.6841 & -0.0294 & -0.2580 \\ -0.6841 & 2.8002 & 0.0113 & 0.0652 \\ -0.0294 & 0.0113 & 0.0005 & 0.0022 \\ -0.2580 & 0.0652 & 0.0022 & 0.0180 \end{bmatrix} > 0$$

and

$$F_1 = \begin{bmatrix} -7.1844 & -0.5599 & 0.0723 & 0.5076 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -10.9127 & -17.6978 & 0.0738 & 0.6304 \end{bmatrix}$$

$$Po_1 = \begin{bmatrix} 0.8831 & 0.0010 & 0.0013 & 0.0161 \\ 0.0010 & 0.8803 & 0.0044 & 0.0764 \\ 0.0013 & 0.0044 & 0.0075 & 0.0565 \\ 0.0161 & 0.0764 & 0.0565 & 0.9589 \end{bmatrix} > 0,$$

$$Po_2 = \begin{bmatrix} 0.2978 & -0.0001 & 0.0006 & 0.0032 \\ -0.0001 & 0.2980 & -0.0010 & -0.0041 \\ 0.0006 & -0.0010 & 0.0214 & 0.0080 \\ 0.0032 & -0.0041 & 0.0080 & 0.2033 \end{bmatrix} > 0$$

and

$$K_1 = \begin{bmatrix} 0.2403 & -0.0001 & 0.0028 & -0.0050 \\ 0.0001 & 0.0000 & 0.0018 & 0.0004 \end{bmatrix}^T,$$

$$K_2 = 10^3 \begin{bmatrix} 0.8766 & 0.0011 & -0.0162 & -0.0045 \\ 0.0008 & 0.0029 & 0.3859 & 0.0644 \end{bmatrix}^T$$

Figures 5.5 and 5.6 show the closed loop behavior of the TORA system with the fuzzy controller and the fuzzy observer, for respectively, the translational position and velocity and the angular position and velocity evolution of the closed loop system for the initial conditions above. The stability of the whole system is checked and the positive real numbers obtained from simulation are:

$\gamma_1 = 0.0027$	$\gamma_2 = 3.1439 \times 10^8$	$\gamma_3 = 2.0702 \times 10^{-6}$	$\gamma_4 = 2.9630 \times 10^8$
$\widetilde{\gamma}_1 = 2.4583 \times 10^8$	$\widetilde{\gamma}_2 = 2.4583 \times 10^8$	$\widetilde{\gamma}_3 = 2.4583 \times 10^8$	$\widetilde{\gamma}_4 = 2.4583 \times 10^8$

Based on a two rules fuzzy model for the TORA system and on only stability conditions without addition of any performance conditions, the obtained results are less conservative and fast, with comparison to the design given in (Tanaka *et al.* , 1998, a), where the TORA system is modelized by four T-S fuzzy rules.

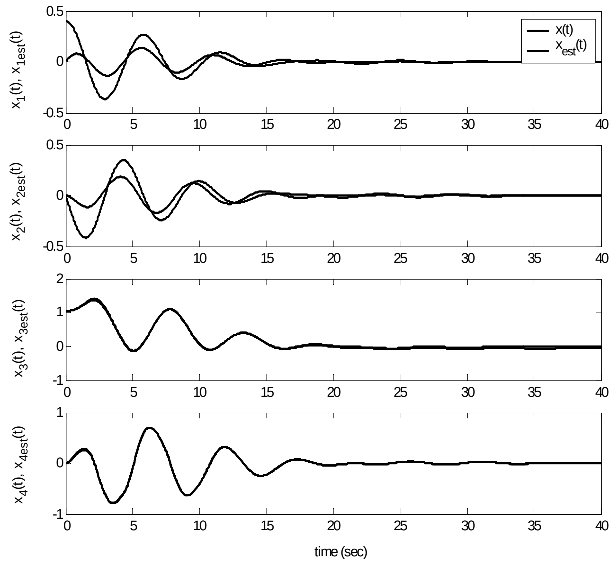

Figure 5.5: TORA performances

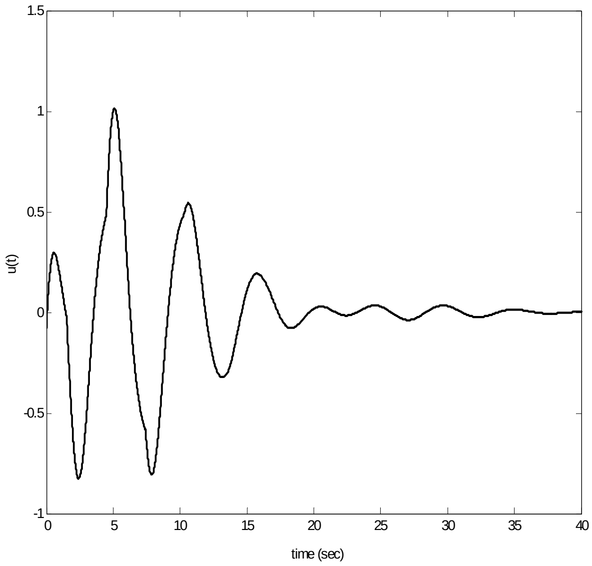

Figure 5.6: TORA input control evolution

5.6 Conclusion

In this chapter, design procedure of fuzzy observer is discussed. The non-quadratic stability conditions also developed in chapter 3 (Abdelmalek *et al.* , 2007), are used for the stabilization of the global T-S fuzzy systems. The controller and the observer are designed separately. The fuzzy controller guarantees the stabilization of the T-S fuzzy system, whereas the fuzzy observer guarantees that the estimation error for states converges to 0. However, to check the stability of the global system i.e. (fuzzy controller+fuzzy observer+fuzzy system), we applied a separation property based on a vector comparison principle proposed by Ma et al. (Ma *et al.* , 1998). The design examples allow us to assess the performances of the proposed observer/controller design and to check the truth of the separation property.

Chapter 6

Conclusion

6.1 Contributions and concluding remarks

The purpose of this thesis is the development of new non-quadratic stability conditions, less conservative, for the control of continuous T-S fuzzy systems, by exploiting the polytopic representation of T-S fuzzy models and using the PDC control design and LMIs.

In this thesis, an outline is given in chapter 2, on fuzzy modeling by two particular structures of fuzzy systems that are Mamdani and T-S fuzzy models, followed by different construction procedures of T-S fuzzy models illustrated by different examples. In the present work, we focused on T-S fuzzy models for their interesting properties that allow us study stability of complex systems. The advantage of T-S fuzzy models is their more powerful capability to represent a complex nonlinear relationship in spite of the smaller number of fuzzy IF-THEN rules. Further, a theorem on the concept of fuzzy systems are universal approximators is given, to show that a fuzzy model is able to approximate and then to represent any real function.

The following step, detailed in chapter 3, concerns the quadratic stability analysis and synthesis ($V(x(t)) = x^T(t)Px(t)$) of T-S fuzzy systems via PDC controllers (Tanaka *et al.* , 1998) and LMIs, one of the important tools in control theory. The main idea of the PDC controller design is to derive each control rule from the corresponding rule of T-S fuzzy model so as to compensate it. The resulting overall fuzzy controller, which is nonlinear in general, is a fuzzy blending of each individual linear controller. Wang et al. (Wang *et al.* , 1996) used the concept of a fuzzy controller that shares the same fuzzy sets with the fuzzy model to design fuzzy controllers to stabilize closed loop fuzzy systems. However, the stability analysis is based on the Lyapunov direct method consisting in finding a quadratic function whose derivative is negative. The used approach seems to be conservative since the $h_i(z(t))$ are not taken into account, it omits all the information contained in the membership functions, further the approach requires to find a common positive definite matrix P for r sub-models at the same time. The inverted pendulum example is studied at the end of this chapter to illustrate the concept of PDC controller design.

In chapter 4 and due to the conservatism of the quadratic stability conditions, we have searched for new stability conditions that realize non-quadratic stabilization and reduce the conservatism, these lead to the results given in (Abdelmalek *et al.* , 2007). We used a fuzzy Lyapunov function $V(x(t)) = \sum_{i=1}^{r} h_i(z(t)) x^T(t) P_i x(t)$, that is a fuzzily blending of quadratic Lyapunov functions and that shares the same membership functions of the fuzzy systems. The conditions were derived in a logical way while profiting fully of the fuzzy Lyapunov function advantage and considering two assumptions that are a proportional relation between the multiple quadratic Lyapunov functions and an upper bound for the time derivative of the premise membership function. The PDC local feedback gains construction procedure is simple and can be solved effectively by optimization computation tools. Our proposal leads to less conservative results and very good results are obtained for different examples, even for those that do not admit a single Lyapunov function and also for T-S systems with several rules (two-links robot with 9 rules) , thus illustrating the effectiveness of the proposed stabilization approach, by reducing conservatism and fast convergence.

The last step in this thesis was the fuzzy observer design to estimate non measurable variables (chapter 5). The non-quadratic stability conditions also developed in chapter 3 (Abdelmalek *et al.* , 2007), were used for the fuzzy observer design. The controller and the observer are designed separately. The fuzzy controller guarantees the stabilization of the T-S fuzzy system, whereas the fuzzy observer guarantees that the estimation error for states converges to 0. The stability of the global system is checked by applying a separation property based on a vector comparison principle proposed by Ma et al. (Ma *et al.* , 1998). The different examples given at the end of chapter 4 allow us to assess the performances of the proposed observer/controller design (Abdelmalek & Goléa, 2009).

6.2 Perspectives and future work

Throughout this thesis, fuzzy control for T-S fuzzy systems stabilization in Lyapunov sense has been discussed and analyzed. We cite here some future possible research direction.

- To extend the proposed stability conditions to uncertain T-S fuzzy systems, and to develop new algorithms including performance constraints in the design of robust fuzzy controllers.

- To extend the proposed approach to the case of observer design in presence of non measurable premise variables.
- To attempt to reduce conservatism by other manners, using other existing control laws or other forms of Lyapunov functions from the literature.

Appendices

Appendix A

Proof of theorem 6 of chapter 3

As is well known from the stability theory, an autonomous dynamical system is stable if there exists a positive definite quadratic function, $V(x(t)) = x^T(t) P x(t)$, which decreases along every nonzero trajectory of the system, and a system having such a Lyapunov function system is called quadratically stable. In the following polytopic system

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) A_i x(t) \tag{A.1}$$

the derivative of V along nonzero trajectory $x(t)$ is given by

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\
&= \left(\sum_{i=1}^{r} h_i(z(t)) A_i x(t)\right)^T P x(t) \\
&\quad + x^T(t) P \left(\sum_{i=1}^{r} h_i(z(t)) A_i x(t)\right) \\
&= \sum_{i=1}^{r} h_i(z(t)) \left[x^T(t) \left(A_i^T P\right) x(t) + x^T(t) (P A_i) x(t)\right] \\
&= \sum_{i=1}^{r} h_i(z(t)) x^T(t) \left\{A_i^T P + P A_i\right\} x(t) < 0 \qquad (A.2)
\end{aligned}$$

since $A_i^T P + P A_i$ is negative when P is positive definite, then the polytopic system (A.1) is quadratically stable if there exists a symmetric matrix P satisfying the following inequalities (Boyd *et al.* , 1994), (Tanaka & Sugeno, 1992):

$$\begin{aligned}
P &> 0, & (A.3) \\
A_i^T P + P A_i &< 0 \quad i = 1, \ ..., r & (A.4)
\end{aligned}$$

Appendix B

LMI transformations for theorem 10 of chapter 3

By multiplying the inequalities (3.17) and (3.18) on the left and right by P^{-1} and defining a new variable $X = P^{-1}$, the conditions are rewritten as

$$-XA_i^T - A_iX + XF_i^TB_i^T + B_iF_iM_i > 0,$$

$$-XA_i^T - A_iX - XA_j^T - A_jX + XF_j^TB_i^T + B_iF_jX + XF_i^TB_j^T + B_jF_iX \geq 0, \quad \text{(B.1)}$$

$$i < j \;\; s.t. \;\; h_i \cap h_j \neq \varnothing$$

Define $M_i = F_iX$ and $X = P^{-1}$, so that for $X > 0$ we have $F_i = M_iX^{-1}$. Substituting into the above inequalities yields

$$-XA_i^T - A_iX + M_i^TB_i^T + B_iM_i > 0, \quad \text{(B.2)}$$

$$-XA_i^T - A_iX - XA_j^T - A_jX + M_j^TB_i^T + B_iM_j + M_i^TB_j^T + B_jM_i \geq 0, \quad \text{(B.3)}$$

$$i < j \;\; s.t. \;\; h_i \cap h_j \neq \varnothing$$

where $X = P^{-1}$and $M_i = F_iX$.

Appendix C

Derivative process of the parameters $\xi_{\rho\ell}$

By using example 4.4.1 of chapter 4 where

$$h_1(x_1(t)) = \frac{1+\sin x_1(t)}{2}, h_2(x_1(t)) = \frac{1-\sin x_1(t)}{2}$$

we have the following equations

$$\begin{aligned}
\frac{\partial h_1(x_1(t))}{\partial x_1(t)} &= \frac{1}{2}\cos x_1(t) = \sum_{\ell=1}^{2} \upsilon_{1\ell}(z(t))\,\xi_{1\ell} \\
\frac{\partial h_2(x_1(t))}{\partial x_1(t)} &= -\frac{1}{2}\cos x_1(t) = \sum_{\ell=1}^{2} \upsilon_{2\ell}(z(t))\,\xi_{2\ell}
\end{aligned}$$

where

$$\sum_{\ell=1}^{2} \upsilon_{1\ell}(z(t)) = 1$$

$$\sum_{\ell=1}^{2} \upsilon_{2\ell}(z(t)) = 1$$

under

$$|x_1(t)| \leq \pi/2$$

we calculate the minimum and the maximum values that correspond to $\xi_{\rho\ell}$

$$\begin{aligned}
\xi_{11} &= \min_{|x_1(t)|\leq\pi/2} \frac{1}{2}\cos x_1\left(t\right) = 0 \\
\xi_{12} &= \max_{|x_1(t)|\leq\pi/2} \frac{1}{2}\cos x_1\left(t\right) = 0.5 \\
\xi_{21} &= \min_{|x_1(t)|\leq\pi/2} \frac{1}{2}\cos x_1\left(t\right) = -0.5 \\
\xi_{22} &= \max_{|x_1(t)|\leq\pi/2} \frac{1}{2}\cos x_1\left(t\right) = 0
\end{aligned}$$

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Résumé

L'objectif de cette thèse est le développement de nouvelles conditions de stabilité de Lyapunov pour les systèmes flous continus du type Takagi-Sugeno (T-S), afin de réduire le degré de conservatisme. Les systèmes non linéaires sont représentés et commandés par une conception à base de systèmes flous T-S. Elle combine la flexibilité de la théorie de logique floue et les outils mathématiques rigoureux d'analyse de la théorie des systèmes linéaires. Les systèmes flous T-S permettent une représentation multimodèles, qui est une forme polytopique convexe. La conception de commande floue la plus utilisée dans la littérature est effectuée en utilisant le concept de la compensation parallèle distribuée (PDC) puisqu'elle partage les mêmes fonctions d'appartenance que les modèles flous T-S. L'idée principale de la conception du contrôleur PDC est de dériver chaque règle de commande à partir de la règle correspondante du modèle flou T-S afin de la compenser. Le contrôleur flou global résultant, qui est non linéaire en général, est une combinaison floue de chaque contrôleur linéaire par retour d'état. L'avantage du modèle flou T-S réside dans le fait que les caractéristiques de stabilité et de performances du système représenté par un modèle flou T-S peuvent être analysées en utilisant l'approche à base de fonction de Lyapunov dont la résolution des conditions de stabilité dépend d'un ensemble d'inégalités matricielles linéaires (LMIs).

Dans cette thèse, de nouvelles conditions de stabilité non-quadratiques sont dérivées. Elles sont basées sur le contrôle PDC pour stabiliser les systèmes flous T-S continus et sur la fonction de Lyapunov floue. Nous obtenons de nouvelles conditions, moins conservatives, qui stabilisent les systèmes flous T-S continus, comprenant aussi ceux qui n'admettent pas une stabilisation quadratique. Notre approche est fondée sur deux hypothèses. La première est basée sur l'existence d'une relation de proportionnalité entre les fonctions de Lyapunov multiples, et la seconde considère une borne supérieure pour la dérivée par rapport au temps de la fonction d'appartenance des prémisses. Les résultats de stabilité obtenus sont étendus au cas où les états ne sont pas disponibles pour la mesure et la rétroaction, en d'autres termes l'observateur flou, en garantissant la stabilité du système global. Cependant, pour vérifier la stabilité du système global, c à d. système flou + contrôleur flou + observateur flou, un principe de séparation est appliqué. Différents exemples sont présentés pour montrer l'efficacité de notre proposition.