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MINISTRE DE L'ENSEIGNEMENT SUPERIEURE ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE ALHADJ LAKHDER - BATNA

Département de Physique

Mémoire :

Présenté par

ZOUIOUECHE MOHAMED

Pour obtenir le diplôme de magister en :

Physique des Rayonnements

Thème

Le phénomène de clustering dans le noyau atomique

Soutenue le : / /2009
Devant le jury:

D.Bahloul	MC.	Université de Batna	Président
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Introduction

The development of physical pictures of atomic nuclei has frequently been accompanied by a steady exchange of ideas with other fields in physics, in particular with atomic, molecular and elementary particle physics. Even hydrodynamics has played an important role in nuclear physics. The ultimate goal of nuclear structure physics is, of course, description of the properties of complex nuclei in terms of interaction between two nucleons.

Current microscopic investigations in nuclear structures theory usually create contact with the shell model of the nucleus. However, In spite of the great successes of this model in the prediction of spins and parities, magic numbers, alpha and beta decay systematic and so forth ,this model still has exhibited many limitations. For example the description of nuclei with large number of valence nucleons.

Collective models have been convenient solution for deformed nuclei. In the 50^{th} the geometric collective model, has been introduced. Its building blocks are shape variables and collective excitations (surface vibration and rotation). However, a link with shell model was clearly established only with the algebraic model called IBM (Interacting Boson Model) developed in the 70^{th}. Similar models have also developed in molecular physics and later applied in nuclear physics. One of the themes is the nuclear vibron model.

In chapter 1 we give a general description of the “classical” structure models: shell model, geometric collective model and IBM. The microscopic cluster model is presented in chapter 2. First we give a detailed equation of the model, then we describe the result of our calculations in the $^{158\text{-}170}$Er framework of this model: energy levels of the ground state bound and the BE(2) transitions. In the last chapter a second approached to clustering is given: the nuclear vibron model. This is an algebraic model resulting from coupling the U(4) and U(6) algebras. Our contribution is deriving an extension of the SU(3) limits of this model by studying the effect of the high order terms for SU(3) Hamiltonian and Electromagnetic transitions.

CHAPTER I:

Overview of nuclear structure

I-1.Introduction

A nucleus is a system of A interacting particles. Besides the general nature of the problem as a many-body system, the nucleon-nucleon interaction is a challenge by itself. Compared to atomic structure, the problem is more complex. In an atom the nucleus provides a common center of attraction for all the electrons (central field) and the electron-electron forces generally play a smaller role. Furthermore, the predominant force (Coulomb) is well known. Nuclei, on the other hand, have no center of attraction; the nucleons are held together by their mutual nuclear interactions which are much more complicated than Coulomb interactions.

The difference between the components of the two systems is another source of difficulties. While all atomic electrons are alike, there are two kinds of nucleons: protons and neutrons. This allows a richer variety of structures. Consequently, there are about 100 types of atoms (elements), but more than 1000 different nuclides.

Experimental evidence for the existence for a nuclear shell structure had already accumulated by the mid 1940's. The early evidence included that nuclei with certain proton and neutron numbers (N or Z = 2, 8, 20, 28, 50, 82, and 126) are particularly stable. These numbers were called "magic numbers" because while the evidence of their special nature was considerable, an understanding of why these numbers were special was lacking.

I-2.The nuclear shell model

I-2-1.Basic principles of the nuclear shell model

The basic assumption of the nuclear shell model is that to a first approximation each nucleon moves independently in a potential that represents the average interaction with the other nucleons in a nucleus. This independent motion can be understood qualitatively from a combination of the weakness of the long-range nuclear attraction and the Pauli exclusion principle.

The complete Schrödinger equation for A nucleons reads as

$$\hat{H}\,\psi(1,2,...,A) = E\,\psi(1,2,...,A) \qquad \text{(I-1)}$$

where $\hat{H}$ contains single nucleon kinetic energies and two-body interactions

$$\hat{H} = \sum_{i=1}^{A}(-\frac{\hbar^2}{2m}\Delta_i) + \sum_{i<j=1}^{A} W(i,j) \tag{I-2}$$

and $\psi(1, 2\ldots, A)$ is a totally antisymmetric wave function, while i denotes all coordinates $\vec{r}_i, \vec{s}_i, \vec{t}_i$ of a given particle ($i = 1, 2,\ldots,A$).

We can add and subtract a potential of the form $\sum_{i=1}^{A} U(i)$ from the Hamiltonian (I-2). We obtain

$$\begin{aligned}\hat{H} &= \sum_{i=1}^{A}(-\frac{\hbar^2}{2m}\Delta_i) + \sum_{i=1}^{A} U(i) + \sum_{i<j=1}^{A} W(i,j) - \sum_{i=1}^{A} U(i) \\ &= \sum_{i=1}^{A}[-\frac{\hbar^2}{2m}\Delta_i + U(i)] + \sum_{i<j=1}^{A} W(i,j) - \sum_{i=1}^{A} U(i) \\ &= \hat{H}^{(0)} + \hat{H}^{(1)}\end{aligned} \tag{I-3}$$

$\hat{H}^{(0)}$ represents a sum of single-particle Hamiltonians, and $\hat{H}^{(1)}$ is represents a residual interaction.

$$\hat{H}^{(0)} = \sum_{i=1}^{A} \hat{h}(i) \tag{I-4-a}$$

$$\hat{H}^{(1)} = \sum_{i<j=1}^{A} W(i,j) - \sum_{i=1}^{A} U(i) \tag{I-4-b}$$

The assumption of the existence of a nuclear average potential allows to hope that there exists such a potential $\sum_{i=1}^{A} U(i)$ and that the residual interaction $\hat{H}^{(1)}$ is small.
There exist a few approaches to deal with this many-body problem.

1. Simple shell model: solve the Schrödinger equation for $\hat{H}^{(0)}$ and neglect the residual interaction $\hat{H}^{(1)}$.

$$\hat{H}^{(0)} \left|\Phi_h\right\rangle = E^{(0)}{}_K \left|\Phi_h\right\rangle \tag{I-5}$$

One suppose that *U(i)* is a known suitable potential (the harmonic oscillator potential, or the Woods-Saxon potential, or the square-well).

2. Realistic shell model: first, one repeats the method given previously; then one takes into account the residual interaction $\hat{H}^{(1)}$ *;* diagonalizing it in the basis $|\Phi_h\rangle$ of the harmonic oscillator potential.

I-2-2.Harmonic Oscillator Potential

In a harmonic oscillator potential, the Hamiltonian is write as.

$$\hat{h} = -\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2 r^2}{2} \tag{I-6}$$

Then the Schrödinger equation

$$\hat{h}\phi(\vec{r}) = \varepsilon\phi(\vec{r}) \tag{I-7}$$

is separable in radial and angular coordinates. The eigenfunctions are given by the products

$$\phi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta,\phi) \tag{I-8}$$

where $Y_{lm}(\theta,\phi)$ are spherical harmonics (they always appear for a spherically symmetric potential). The radial wave functions for the harmonic oscillator potential are given by [1]

$$R_{nl}(r) = N_{nl} r^l \exp\left(-\frac{r^2}{2b^2}\right) L_{n-1}^{\,l+1/2}\left(\frac{r^2}{b^2}\right) \tag{I-9}$$

where $b=(\hbar/m\omega)^{1/2}$ and $L_{n-1}^{\,l+1/2}(x)$ are Laguerre polynomials (they are tabulated for given n and l) .The normalization factor N_{nl} is defined by the condition

$$\int_0^\infty r^2 R_{nl}^{\,2}(r)dr = 1 \tag{I-10}$$

Then Laguerre polynomials satisfy the following relation

$$\int_0^\infty e^{-r} r^{l-1/2} L_{n-1}^{\,l+1/2}(r)\cdot L_{n-1}^{\,l+1/2}(r)\,dr = \frac{1}{l+1/2}\frac{\left((n-1)!\right)^3}{\left(n-l-3/2\right)}$$

(I-11)

The energy eigenvalues are given by

$$\varepsilon_N = \hbar\omega\left(2(n-1)+l-\frac{1}{2}\right)$$
$$=\hbar\omega\left(2N+\frac{3}{2}\right) \quad \text{(I-12)}$$

$$N = 0,1,2,\ldots, \quad \text{(I-13-a)}$$
$$l = N,N\text{-}2,\ldots,1 \text{ or } 0 \quad \text{(I-13-b)}$$
$$n = (N\text{-}l\text{+}2)/2$$

The energy level with a given N is called an oscillator shell. The resulting levels can be denoted as

$N = 0$ $1s$
$N = 1$ $1p$
$N = 2$ $1d,2s$
$N = 3$ $1f,2p$
$N = 4$ $1g,2d,3s$

(here the numbers refer to n and the letters denote l).

Applying the parity operation $\hat{P}$ on the eigenfunctions gives

$$\hat{P}\phi_{nlm}(\vec{r}) = \hat{P}(R_{nl}(r)Y_{lm}(\theta,\phi))$$
$$= (R_{nl}(r)\hat{P}Y_{lm}(\theta,\phi)$$
$$= R_{nl}(r)(-1)^l Y_{lm}(\theta,\phi) \quad \text{(I-14)}$$
$$= (\text{-})^l\phi_{nlm}(\vec{r})$$

Each oscillator shell contains orbitals with either even or odd l and it is either even or odd with respect to $\hat{P}(\vec{r}\rightarrow-\vec{r})$

The total degeneracy of the N th oscillator shell is $2(N+1)(N+2)$

One takes the intrinsic spin s = 1/2 and isospin t = 1/2 of nucleons, and having

$$\hbar\omega = 41A^{-1/3}\ \text{MeV} \quad \text{(I-15)}$$

Adding the spin-orbit coupling term [1]:

$$\hat{h} = -\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2 r^2}{2} + f(r)(\vec{l}.\vec{s}) \tag{I-16}$$

Taking into account the intrinsic spin of the nucleons, we can write down the single-particle wave functions as

$$\phi_{nlsjm}(\vec{r},\vec{s}) = R_{nl}(r)\left[Y_l(\theta,\phi)\times\chi_{1/2}(\vec{s})\right]_m^{(j)} \tag{I-17}$$

where the orbital and the spin angular momenta are coupled to a total angular momentum *j*. These wave functions are eigenfunctions of the Hamiltonian (I-16). Taking into account the isospin of the nucleons, the final single-particle wave functions are

$$\phi_{nlsjm,tmt}(\vec{r},\vec{s},\vec{t}) = R_{nl}(r)\left[Y_l(\theta,\phi)\times\chi_{1/2}(\vec{s})\right]_m^{(j)}\theta_{1/2}(\vec{t}) \tag{I-18}$$

To get the value of the spin-orbit splitting we will calculate the matrix elements

$$\Delta\varepsilon_{nlsjm} = \langle nlsjm|\, f(r)(\vec{l}.\vec{s})\,|nlsjm\rangle = \begin{cases} +1/2l\langle f(r)\rangle_{\rm nl} & \text{for} \quad j = l+1/2 \\ -1/2(l+1)\langle f(r)\rangle_{nl} & \text{for} \quad j = l-1/2 \end{cases} \tag{I-19}$$

Here

$$\langle f(r)\rangle_{nl} = \langle nl|\, f(r)\,|nl\rangle = \int R_{nl}^{\,*} R_{nl}\, f(r) r^2 dr \tag{I-20}$$

Each oscillator shell splits into orbitals:

$$\begin{aligned} &N=0 \quad 1s_{1/2} \\ &N=1 \quad 1p_{1/2},\ 1p_{3/2} \\ &N=2 \quad 1d_{5/2},\ 1d_{3/2},\ 2s_{1/2} \\ &N=3 \quad 1f_{7/2},\ 1f_{5/2},\ 2p_{3/2},\ 2p_{1/2} \\ &N=4 \quad 1g_{9/2},\ 1g_{7/2},\ 2d_{5/2},\ 2d_{3/2},\ 3s_{1/2} \end{aligned} \tag{I-21}$$

I-2-3.Woods-Saxon Potential

A more realistic description of the single-particle levels is given by the Woods-Saxon potential, where the depth of the well is proportional to the density of nucleons, which is given for the spherical case by,

$$\rho(r) = \frac{\rho_0}{1+\exp\left[\dfrac{r-R}{a}\right]} \tag{I-22}$$

where *r* is the distance of the nucleons with respect to the centre of the nucleus, ρ_0 is a constant, *R* is the radius of the nucleus and *a* is the surface thickness. The spherical Woods-Saxon potential is therefore taken as [2],

$$V = -\frac{V_0}{1+\exp\left[\frac{r-R}{a}\right]} \tag{I-23}$$

Typical values for the parameters are: depth $V^0 \approx 50$MeV, radius $R \approx 1.1A^{1/3}$ fm, and surface thickness a≈0.5 fm.

Addition of the spin-orbit interaction to the Hamiltonian is necessary to produce magic numbers [3]

$$\hat{h} = -\frac{\hbar^2}{2m}\Delta + \frac{U_0}{1+\exp\left[\frac{r-R}{a}\right]} + f(r)(\vec{l}.\vec{s}) \tag{I-24}$$

We can to generalize the spherical Woods-Saxon potential to deformed nuclei by the following,

$$V(r,\beta_2) = -\frac{V_0}{1+\exp\left[\frac{r-R(r,\beta_2)}{a}\right]} \tag{I-25}$$

where (r,β_2) denotes a set of all the shape parameters uniquely specifying the nuclear surface. In the full Hamiltonian, a spin-orbit term must be added to the Woods-Saxon potential in order to reproduce the correct magic numbers. This is given by,

$$V_{S.O}(r,\beta_2) = -\lambda\left(\frac{\hbar}{2mc}\right)^2\left[\vec{\nabla}V(r,\beta_2)_{S.O}\times\vec{P}\right].\vec{s} \tag{I-26}$$

I-2-4.Shell model with pure configurations

In pure configuration we neglect the residual interaction $\hat{H}^{(1)}$. The solution of the Schrödinger equation with a Hamiltonian $H^{(0)}$ (I-4) can be given by

$$\varphi_{k1}(1)\,\varphi_{k2}(2)\,\ldots\,\varphi_{kA}(A) \tag{I-27}$$

Here each k_i labels the single-particle state $|nlsjm,tm_t>$, while (i) refers to all coordinates of a nucleon, $i \equiv (\vec{r}_i, \vec{s}_i, \vec{t}_i)$. The single-particle wave functions are solutions of the corresponding Schrödinger equations

$$\hat{h}(i)\varphi_k(i) = \varepsilon_k \varphi_k(i) \tag{I-28}$$

This can be a single-particle equation with a harmonic oscillator potential, (I-16), or with Woods-Saxon potential, (I-23).

However, since dealing with the protons and neutrons, the correct shell model wave function should be antisymmetric under permutation of any two nucleons with respect to its space, spin and isospin coordinates and it should have definite values of the total angular momentum J and the total isospin T. So, in constructing the final shell model wave functions as linear combinations of functions (I-27), totally antisymmetric and coupled to J and T. indicating them as

$$\Phi_I(1,2,\ldots,A) \tag{I-29}$$

where $\Gamma=(J,T)$

$$\hat{H}^{(0)}\Phi_I(1,2,\ldots,A)=E^{(0)}_{\Gamma}\,\Phi_I(1,2,\ldots,A) \tag{I-30}$$

The total energy is thus given by

$$E_{\Gamma}^{(0)} = \sum_{i=1}^{i=A} \varepsilon_{ki} \tag{I-31}$$

The residual interaction is then added as a perturbation.

I-2-5.Shell model with configuration mixing:

In the configuration mixing, $\hat{H}^{(1)}$ is taken into account. One solves the Schrödinger equation

$$\hat{H}\left|\psi_p\right\rangle = E_p\left|\psi_p\right\rangle \tag{I-32}$$

In this case, solutions of $\hat{H}^{(0)}$ (functions Φ_Γ (I-29)) are considered only as a basis for the diagonalization of the full Hamiltonian $\hat{H}$. Thus, the wave function of the system is expanded as

$$\left|\psi_p\right\rangle = \sum_{i=1}^{g} a_{kp}\left|\Phi_k\right\rangle \tag{I-33}$$

where g denotes the number of pure configurations $|\Phi_k\rangle$ considered; it is related to the model space used. Usually, the model space incorporates all possible configurations of N_π valence protons and N_ν valence neutrons in the partially filled orbitals, while the rest is considered as an inert core. Since the Hamiltonian $\hat{H}$ is invariant in the space and isospace, its eigenstates are characterized by total angular momentum J and isospin T. Substituting (I-33) into equation (I-32) [3], we get

$$(\hat{H}^{(0)} + \hat{H}^{(1)})\sum_{k=1}^{g} a_{kp}\left|\Phi_k\right\rangle = E_P \sum_{k=1}^{g} a_{kp}\left|\Phi_k\right\rangle \tag{I-34}$$

Since

$$\hat{H}^{(0)}\left|\Phi_k\right\rangle = E^{(0)}{}_k\left|\Phi_k\right\rangle \tag{I-35}$$

the matrix elements of the Hamiltonian $\hat{H}$ are given by

$$H_{lk} \equiv \left\langle \Phi_l \left| \hat{H} \right| \Phi_K \right\rangle = E_k{}^{(0)}\delta_{lk} + H_{lk}{}^{(1)} \tag{I-36}$$

where

$$H_{lk}{}^{(1)} = \left\langle \Phi_l \left| \hat{H}^{(1)} \right| \Phi_k \right\rangle \tag{I-37}$$

Thus we have to solve a system of equations

$$\sum_{k=1}^{g} H_{lk} a_{kp} = E_p a_{lp} \tag{I-38}$$

that means to diagonalize the matrix H_{lk} and to find the eigenvalues E_p and the coefficients a_{kp}. Since the basis is orthogonal and normalized, the eigenvectors belonging to different eigenvalues are necessarily orthogonal and can be normalized such that

$$\sum_{k=1}^{g} a_{kp} a_{kp'} = \delta_{pp'} \qquad \text{for} \qquad E_p \neq E_{p'} \tag{I-39}$$

Equation (I-38) can be re-written as

$$\sum_{l,k=1}^{g} a_{lp'} H_{lh} a_{kp} = E_p \delta_{pp'} \tag{I-40}$$

or in a matrix form

$$A^{-1}HA = E \tag{I-41}$$

where on the right-hand side is a diagonal matrix.

I-3.Geometric Collective Model

I-3-1.Spherical nucleus

I-3-1-1.Quantisation of surface vibrations

The collective coordinates describing the nuclear surface motion are defined by the expansion of the surface into spherical harmonic

$$R(\vartheta,\varphi,t) = R_0\left[1+\sum_{\lambda u}(-1)^u \alpha_{\lambda -u}(t)Y_{\lambda u}(\vartheta,\varphi)\right] \qquad \text{(I-42)}$$

$\alpha_{\lambda u}$ are the components of an irreducible tensor of rank λ with u=λ, λ-1, λ-2,…, -λ .

Using the property $Y_{\lambda u}{}^*(\vartheta,\varphi) = (-)^u Y_{\lambda -u}(\vartheta,\varphi)$ one gets

$$\alpha_{\lambda u}{}^* = (-)^u \alpha_{\lambda -u} \qquad \text{(I-43)}$$

Low-lying excitations correspond to small oscillations around a spherical equilibrium shape. They are described by the harmonic oscillator Hamiltonian

$$\hat{H} = T + V = \frac{1}{2}\sum_{\lambda u}\left(B_\lambda \left|\dot{\alpha}_{\lambda u}\right|^2 + C_\lambda \left|\alpha_{\lambda u}\right|^2 \right) \qquad \text{(I-44)}$$

The parameters of inertia B_λ and stiffness C_λ are real constants that can be calculated within the fluid picture.

The second quantized form of $\hat{H}$ can be obtained by introducing bose operators $b_{\lambda u}$ (and their hermitic conjugates). They are related to the coordinates $\alpha_{\lambda u}$ and corresponding momenta $\pi_{\lambda u}$ by

$$\alpha_{\lambda u} = \left(\frac{\hbar}{2B_\lambda \omega_\lambda}\right)^{1/2}\left(b_{\lambda u}{}^+ + (-)^u b_{\lambda -u}\right) \qquad \text{(I-45-a)}$$

$$\pi_{\lambda u} = i\left(\frac{\hbar}{2}B_\lambda \omega_\lambda\right)^{1/2}\left((-)^u b_{\lambda -u}{}^+ - b_{\lambda u}\right) \qquad \text{(I-45-b)}$$

The operators $b_{\lambda u}$ obey Bose commutation rules

$$\left[b_{\lambda u}, b_{\lambda' u'}\right] = 0 \qquad \text{(I-46-a)}$$

$$\left[b_{\lambda u}, b_{\lambda' u'}{}^+\right] = \delta_{\lambda\lambda'}\delta_{uu'} \qquad \text{(I-46-b)}$$

The harmonic Hamiltonian becomes:

$$\hat{H} = \frac{1}{2}\sum_{\lambda u}\hbar\omega_{\lambda}\left(b_{\lambda u}{}^{+}b_{\lambda u} + \frac{1}{2}\right) \tag{I-47}$$

with the frequency

$$\omega_{\lambda} = \sqrt{\frac{C_{\lambda}}{B_{\lambda}}} \tag{I-48}$$

The operators $b^{+}_{\lambda u}$ create phonons with angular momentum λ and magnetic quantum number *u*.

One introduces the phonon number operator

$$\hat{N}_{\lambda u} = b_{\lambda u}{}^{+}b_{\lambda u} \tag{I-49}$$

This operator commutes with the harmonic oscillator Hamiltonian

$$\left[\hat{H}, \hat{N}_{\lambda u}\right] = 0 \tag{I-50}$$

The harmonic oscillator Hamiltonian can be expressed in terms of $N_{\lambda u}$ as:

$$\hat{H} = \frac{1}{2}\sum_{\lambda u}\hbar\omega_{\lambda}\left(\hat{N}_{\lambda u} + \frac{1}{2}\right) \tag{I-51}$$

Then

$$\hat{H} = \frac{1}{2}\sum_{\lambda}\hbar\omega_{\lambda}\left(\hat{N}_{\lambda} + \frac{2\lambda+1}{2}\right) \tag{I-52}$$

The corresponding energy is

$$E = \frac{1}{2}\sum_{\lambda}\hbar\omega_{\lambda}\left(N_{\lambda} + \frac{2\lambda+1}{2}\right) \tag{I-53}$$

Each phonon carries the energy quantum $\hbar\omega_{\lambda}$, has an angular momentum is λ and with projection *u* and its parity is $\pi = (-1)^{\lambda}$. The lowest-lying states are as follows.

To determine the spectrum and the various states, we define a phonon vacuum

$$b_{\lambda u}\left|0\right\rangle = 0 \tag{I-54}$$

And hence

$$N_{\lambda u}\left|0\right\rangle = 0 \tag{I-55}$$

The one-boson states is defined by

$$\left|\lambda u\right\rangle = b^{+}_{\lambda u}\left|0\right\rangle \tag{I-56}$$

The two-bosons states are easily constructed for a total angular momentum *L* as follows

$$\left|N=2\ \ LM\right\rangle=\sum_{u1u2}\left\langle\lambda_1 u_1\lambda_2 u_2\left|LM\right\rangle b^{+}_{\lambda 1u1}b^{+}_{\lambda 2u2}\right|0\right\rangle \qquad \text{(I-57)}$$

The most general eigenstate can be labelled by $|NLM>$, N being the number of phonons.

I-3-1-2.Different vibration modes

Each value of λ corresponds to a vibration mode characterised with 2^{λ}. For example

λ=2 → 2^2 quadrupole mode

λ=3 → 2^3 octupole mode

λ=4 → 2^4 hexadecapole mode

i) Quadrupole (λ=2) and octupole (λ=3) modes

The classical quadrupole vibration Hamiltonian can be written as

$$\hat{H}=\frac{1}{2B_2}\sum_{u=-2}^{u=2}\left|\pi_{\lambda u}\right|^2+\frac{C_2}{2}\sum_{u=-2}^{u=2}\left|\alpha_{\lambda u}\right|^2 \qquad \text{(I-58)}$$

Equation (I-52) reduces to

$$\hat{H}=\frac{1}{2}\hbar\omega_2\left(\hat{N}_2+\frac{5}{2}\right) \qquad \text{(I-59)}$$

and the energies of the states will be given by

$$E=\frac{1}{2}\hbar\omega_2\left(N_2+\frac{5}{2}\right) \qquad \text{(I-60)}$$

The corresponding spectrum is shown in figure I-1. The octupole vibration spectrum can be obtained in a similar way.

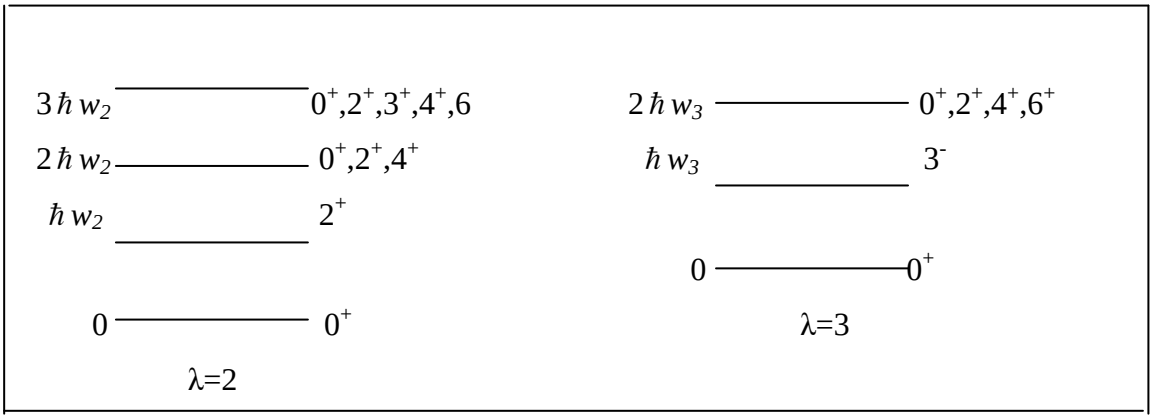


Figure I-1 Harmonic spectrum for the quadrupole (λ=2) and octupole (λ=3) modes

I-3-2.Deformed nucleus

I-3-2-1. Bohr Hamiltonian

The rotation-vibration model treats nuclei that have axially symmetric deformed equilibrium configuration in the collective potential energy. The simplest nuclear collective quadrupole Hamiltonian has the form

$$\hat{H} = \frac{1}{2} B_2 \sum_u \dot{\alpha}_{2u}^{*} \dot{\alpha}_{2u} + V(\alpha_{2u}) \tag{I-61}$$

The collective Kinetic energy in the laboratory system is given to lowest order in the collective variables by

$$\hat{T} = \frac{1}{2} B_2 \sum_u \dot{\alpha}_{2u}^{*} \dot{\alpha}_{2u} \tag{I-62}$$

It has to be transformed into the intrinsic coordinates by using the relations

$$\alpha_{2u} = \sum_v D^2_{uv} {}^{*}(\vartheta_1, \vartheta_2, \vartheta_3) a_{2u} \tag{I-63-a}$$

$$a_{2u} = \sum_v D^2_{uv}(\vartheta_1, \vartheta_2, \vartheta_3) \alpha_{2u} \tag{I-63-b}$$

The kinetic energy should be expressed in terms of the angular momenta about the body-fixed axes, because only in this way will the moments of inertia be constant during the rotation.

In the intrinsic coordinates one can show that

$$a_1 = a_{-1} = 0 \tag{I-64-a}$$

$$a_2 = a_{-2} \tag{I-64-b}$$

Hill and Wheeler introduced the two shape defining parameters β and γ by

$$a_0 = \beta \cos\gamma \tag{I-65-a}$$

$$a_2 = \frac{\sqrt{2}}{2} \beta \sin\gamma \tag{I-65-b}$$

The total kinetic energy is

$$\hat{T} = \frac{1}{2} \sum_{\substack{k,k' \\ v,\sigma}} \omega_k {}^{'2} \, g_k(a_0, a_2) + \sum_v \dot{a}_v {}^{*} \dot{a}_{-v} \tag{I-66}$$

where J_i is the moment of inertia which is given by

$$J_i = 4B\beta^2 \sin^2(\gamma - \frac{2k\pi}{3}) \tag{I-67}$$

The collective potential energy surface as a function of the quadrupole coordinates has the form

$$\begin{aligned} V(\alpha^{[2]}) = \frac{1}{2}\sqrt{5}C_2\left[\alpha^{[2]}\times\alpha^{[2]}\right]^{(0)} + C_3\left[\left[\alpha^{[2]}\times\alpha^{[2]}\right]^{(2)}\times\alpha^{[2]}\right]^{(0)} \\ + C_4\left[\alpha^{[2]}\times\alpha^{[2]}\right]^{(0)}.\left[\alpha^{[2]}\times\alpha^{[2]}\right]^{(0)} + ... \end{aligned} \tag{I-68}$$

If we express $V(\alpha^{[2]})$ in terms of Euler angles and intrinsic coordinates we obtain

$$\begin{aligned} V(a_0, a_2) = \frac{1}{2}\sqrt{5}C_2\left[a^{[2]}\times a^{[2]}\right]^{(0)} + C_3\left[\left[a^{[2]}\times a^{[2]}\right]^{(2)}\times a^{[2]}\right]^{(0)} \\ + C_4\left[a^{[2]}\times\alpha^{[2]}\right]^{(0)}.\left[a^{[2]}\times a^{[2]}\right]^{(0)} + ... \end{aligned} \tag{I-69}$$

This reduces to

$$\begin{aligned} V(a_0, a_2) = \frac{1}{2}C_2(a_0{}^2 + 2a_2{}^2) + \sqrt{\frac{2}{35}}C_3 a_0(6a_2{}^2 + 2a_0{}^2) \\ + \frac{1}{5}C_4(a_0{}^2 + 2a_2{}^2)^2 + ... \end{aligned} \tag{I-70}$$

For well deformed nuclei one can expand the potential around the equilibrium position (β_0, γ_0) to get

$$V(\beta,\gamma) = V(\beta_0,\gamma_0) + \frac{1}{2}C_\beta(\beta-\beta_0)^2 + \frac{1}{2}C_\gamma(\gamma-\gamma_0)^2 \tag{I-71}$$

The kinetic energy is taken in terms of Euler angles and intrinsic variables

$$\hat{T} = \frac{1}{2}\sum_k \frac{I'^2_k}{2J_k(a_\nu)} + \frac{1}{2}B(\dot{a}_0{}^2 + 2\dot{a}_2{}^2) \tag{I-72}$$

Finally the quantized form of the Hamiltonian (Bohr Hamiltonian) can be written as

$$\hat{H} = \sum_k \frac{I_k'^2}{J_k(\beta,\gamma)} - \frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial}\right] + V(\beta,\gamma) \tag{I-73}$$

In order to decouple rotation and vibration one can expand the moments of inertia as follows

$$\frac{1}{J_i(\beta,\gamma)} = \frac{1}{J_i(\beta_0,\gamma_0)} + \frac{1}{1!}\left(\frac{\partial}{\partial\beta}\frac{1}{J_i(\beta,\gamma)}\right)_{\beta_0,\gamma_0}(\beta-\beta_0) + \frac{1}{1!}\left(\frac{\partial}{\partial\gamma}\frac{1}{J_i(\beta,\gamma)}\right)_{\beta_0,\gamma_0}(\gamma-\gamma_0) + \ldots \quad \text{(I-74)}$$

The Hamiltonian of the rotation-vibration model separates into three terms

$$\hat{H} = \hat{H}_{\text{rot}} + \hat{H}_{\text{vib}} + \hat{H}_{\text{vib rot}} \quad \text{(I-75)}$$

The eigenfunctions of I^2, I_3, I_3' are given by

$$\left|IMK\right\rangle = \sqrt{\frac{2I+1}{8\pi^2}} D^I_{MK}{}^*(\vartheta_1,\vartheta_2,\vartheta_3) \quad \text{(I-76)}$$

Where $I^2$is the total angular momentum and its components $I_3 = M$ and $I_3' = K$

Since H (see Eq.(I-71)), I^2, and I_3, commute, the eigenfunctions of collective Hamiltonian Eq.(I-73) have the general form

$$\left|\psi^I_M\right\rangle = \sum_k g_k(\beta,\gamma)\left|IMK\right\rangle \quad \text{(I-75)}$$

I is the total angular momentum and M is the projections of I onto the laboratory z axis and K is the projection of I onto body-fixed 3-axis.

I-3-2-2.Solution of the rotation-vibration Hamiltonian

In the Hamiltonian of the rotation-vibration model, which is given by Eq(I-72) ,the $\hat{H}_{\text{vib rot}}$ is a small perturbation; one can treat it either in perturbation theory or by diagonalization.

I-3-2-2-1.The axially symmetric case

The Hamiltonian of an axially symmetric well deformed nucleus is

$$H = \frac{I'^2 - I'_3{}^2}{2J_0} + \frac{1}{2}C_\beta(\beta-\beta_0)^2 + \frac{1}{2}C_\gamma(\gamma-\gamma_0)^2 \quad \text{(I-76)}$$

The eigenstates are symmetrized using the rotation operators R_1 and R_2 to get

$$\left|\psi^I_{MK}\right\rangle = g_K(\beta,\gamma)\left\{\left|IMK\right\rangle + (-)^I\left|IM(-K)\right\rangle\right\} \quad \text{(I-77)}$$

where K an even integer and

$$I = \begin{cases} I = 0,2,4,\ldots \text{ for } K = 0 \\ I = K, K+1, K+2, \ldots \text{ for } K \neq 0 \end{cases} \quad \text{(I-78)}$$

The total energy associated with total wave functions, is now given by

$$E_{I,K,n_\beta,n_\gamma} = \hbar\omega_\beta(n_\beta + \frac{1}{2}) + \hbar\omega_\gamma(2n_\gamma + \frac{1}{2}|K| + 1) + \frac{\hbar^2(I(I+1) - K^2)}{2J_0} \quad \text{(I-79)}$$

where

$$n_\beta = 0,1,2,\ldots \text{and} \quad n_\gamma = 0, 1, 2,\ldots, \quad \text{(I-80)}$$

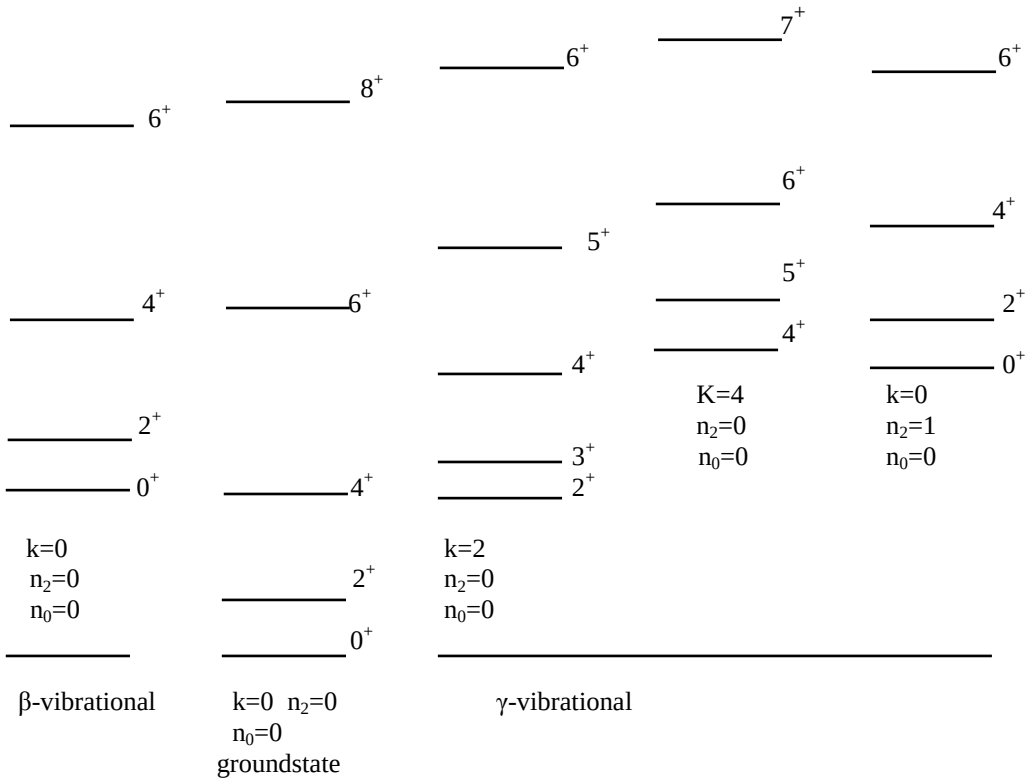
The bands of the spectrum fig (I-2) are characterized by a given set of (K, n_β , n_γ) and follow the $I(I + 1)$ rule of the rigid rotor . The principal bands are [4]:

1-The ground-state band, made up of the states $|IM000\rangle$ with I even.

2-The β band, containing the states $|IM010\rangle$ with one quantum of vibration added in the β direction and lies at energy $\hbar\omega = \hbar(C_0 / B)^{1/2} = E_\beta$ above the ground states.

3-The γ–vibration band, is described $|IM200\rangle$

4- The next higher bands. These should be the additional γ bands with $K = 4$ and the one with $n_\gamma = 1$.



FigureI-2 Typical band structure for a deformed even-even nucleus

I-3-2-2-2.The Asymmetric Rotor

An interesting spectrum is exhibited by the asymmetric rotor, which was first investigated by Davydov and Filippov. This model assumes a rotational Hamiltonian with all three different moments of inertia.

$$\hat{H}_{\text{rot}} = \sum_k \frac{I'^2_k}{2J_k(a_v)} \tag{I-81}$$

Through the inserting the moments of inertia form Eq.(I-67),one can rewrite this Hamiltonian in the form

$$\hat{H}_{\text{rot}} = \frac{1}{4B\beta^2} \sum_k \frac{I'^2_k}{\sin^2(\gamma - \frac{2}{3}k\pi)} \tag{I-82}$$

$\hat{H}_{\text{rot}}$ splits into the Hamiltonian for a symmetric rotor plus a remaining term

$$\begin{aligned} \hat{T}_{\text{rot}} = (I'^2 - I'^2_3)(\frac{1}{4g_1} + \frac{1}{4g_2}) + \frac{I'^2_3}{2g_3} \\ + (I'^2_1 - I'^2_2)(\frac{1}{4g_1} - \frac{1}{4g_2}) \end{aligned} \tag{I-83}$$

The solution of the symmetric rotator which are symmetrized under R_1 and R_2 are given by

$$\left| IMK \right\rangle = \sqrt{\frac{(2I+1)}{16\pi^2(1+\delta_{K0})}} (D^I_{M-K} * (\vartheta_j) + (-1)^{I-2K} D^I_{M-K} * (\vartheta_j)) \tag{I-84}$$

where

$$K=0,2,4,\ldots, \tag{I-85-a}$$

$$\text{For} \quad k\neq 0 \quad I=K, K+1, K+2,\ldots \tag{I-85-b}$$

$$\text{For} \quad K=0 \quad I=0,2,4,\ldots \tag{I-85-c}$$

Then the third term in Eq(I-83) must be diagonlized in the basis set of Eq(I-84). The most general solution thus has the form

$$\left| \psi_{IMi} \right\rangle = \sum_k A^I_K(\gamma) \left| IMK \right\rangle \tag{I-86}$$

and

$$K = 0, 2, 4,\ldots, \tag{I-87}$$

The index i on the wave function ψ_{IMi} indicates that in general several states of given spin I may occur.

I-4.Algebraic Collective Model

I-4-1.Introduction

The interacting boson model (IBM) has been broadened and developed steadily since its inception (Arima and Iachello 1975) some three decades ago so that, by now, it consists of a family of nuclear structure models linked by common underlying assumptions concerning their microscopic foundation and by the algebraic origin of their formalism.

Showing that the IBM-Hamiltonian is composed of elements which constitute a Lie algebra. And illustrate the subalgebras are looked into. Three chains of subalgebras can be formulated. Furthermore are treated special cases of the IBM.

I-4-2.The U(6) algebra

In the original version of the IBM, applicable to even-even nuclei, the shell model reveals that the low-lying states of the even-even nuclei are made up predominantly by nucleon pairs with total spin 0 or 2. Thus, the basic building blocks are *s* and *d* bosons interacting via two-body forces, hence the name interacting boson model. This model has the group structure of U(6).

The U(6) algebra may be realized in terms of six creation and six annihilation operators b_i^+ and b_i (i=1,2,…,6), where one uses the notation

$$b_n^+ = d_{n-3}^+ \quad \text{for} \quad n=1,\ldots,5 \quad \text{and} \quad b_6^+ \equiv s^+ \tag{I-88-a}$$

$$b_n = d_{n-3} \quad \text{for} \quad n=1,\ldots,5 \quad \text{and} \quad b_6 = s \tag{I-88-b}$$

Writing the commutation rules for boson operators especially for *s*- and *d* operators:

$$[d_\upsilon , d^+_\nu]=\delta_{\nu\upsilon} \quad \text{and} \quad [s, s^+]=1 \tag{I-89}$$

Lie algebra associated with this structure is generated from the 36 bilinear operators

$$G_i^j = b_i^+ b_j \tag{I-90}$$

The bilinear operator is satisfy the *U*(6) commutation relation

$$[G_i^j , G_k^l] = G_i^l \delta_{jk} - G_k^j \delta_{il} \tag{I-91}$$

One can also define the generator of this algebra in coupled form as

$$\begin{aligned} B_u^{(\lambda)}(l,l') &= [b_l^+ \times \tilde{b}_{l'}]_u^{(\lambda)} \\ &= \sum_{mm'} \langle lm\, l'm' | \lambda u \rangle b_{lm}^+ b_{l'm'} \end{aligned} \tag{I-92}$$

I-4-3- Chaines of subalgebra of U(6)

Using the canonical group chains

$$SU(n) \supset SU(n-1) \supset ... \supset SU(1) \tag{I-93-a}$$

$$SO(n) \supset SO(n-1) \supset ... \supset SO(2) \tag{I-93-b}$$

$$U(n) \supset SU(n) \tag{I-93-c}$$

$$U(n) \supset SO(n) \tag{I-93-d}$$

One can show that $U(6)$ admits three chains of subalgebra. They are obtained as follows [7].

a- Removing the 11 operators containing *s*-boson from $U(6)$ algebra we obtain the algebra of $U(5)$ which is generated by the 25 operators of the form $[d^+ \times \tilde{d}]_m^{(l)}$. Furthermore, after deleting the 15 operators of the form $[d^+ \times \tilde{d}]_m^{(l)}$ with l = 0, 2, 4, we obtain 20 operators generate the group $SO(5)$. Finally we drop the 7 operators of the form $[d^+ \times \tilde{d}]_m^{(3)}$, we are then left with the angular momentum operators $[d^+ \times \tilde{d}]_m^1$ which generate $SO(3)$ .we have thus found the first IBM chain

$$u(6) \supset u(5) \supset so(5) \supset so(3) \tag{I-94}$$

b- The operators $[d^+ \times \tilde{d}]_m^{(1)}$ and $\left[d^+ \times s\right]^{(2)}{}_m + \left[s^+ \times \tilde{d}\right]_m^{(2)} - (\sqrt{7}/2)\left[d^+ \times \tilde{d}\right]_m^{(2)}$ constitute a Lie algebra with the dimension n = 3 + 5= 8, which is in fact the $SU(3)$. Since $[d^+ \times \tilde{d}]_m^{(1)}$ are the generators of so(3) we obtain the second group chain:

$$u(6) \supset su(3) \supset so(3) \tag{I-95}$$

c- The elements of $[d^+ \times \tilde{d}]_m^{(l)}$ (l=1,3) and $\left[d^+ \times s\right]_u^{(2)} + \left[s^+ \times \tilde{d}\right]_m^{(2)}$ constitute a Lie algebra of n =3 + 7 + 5 = 15 dimensions ,which is the $SO(6)$ algebra. By applying the $SO(n) \supset SO(n-1)$ reduction for n = 6 one gets the third IBM chain

$$u(6) \supset so(6) \supset so(5) \supset so(3) \tag{I-96}$$

I-4-4.The IBM Hamiltonian

The general Hamiltonian in the *s*-*d* boson space for one and two-body interactions is [6]

$$\hat{H} = E_0 + \sum_{ij} e_{ij} bb + \sum_{ijkl} e_{ijkl} b_i^+ b_j^+ b_l b_k \tag{I-97}$$

where E_0 is a constant and e_{ij} and e_{ijkl} are parameters that define the single-boson energies and the interaction between the bosons.

After imposing the hermiticity condition and rotation invariant of the Hamiltonian, the Hamiltonian (I-97) then reduces to [6]

$$\begin{aligned}\hat{H} = E_0 + e_s \hat{n}_s + e_d \hat{n}_d + e_1 \left[\left[d^+ \times d^+ \right]^{(0)} \times \left[\tilde{d} \times \tilde{d} \right]^{(0)} \right]_0^{(0)} \\ + e_2 \left[\left[d^+ \times d^+ \right]^{(2)} \times \left[\tilde{d} \times \tilde{d} \right]^{(2)} \right]_0^{(0)} + \mathrm{e}_3 \left[\left[d^+ \times d^+ \right]^{(4)} \times \left[\tilde{d} \times \tilde{d} \right]^{(4)} \right]_0^{(0)} \\ + e_4 \left[\left[d^+ \times d^+ \right]^{(0)} \times \left[\tilde{s} \times \tilde{s} \right]^{(0)} + \left[s^+ \times s^+ \right]^{(0)} \times \left[\tilde{d} \times \tilde{d} \right]^{(0)} \right]_0^{(0)} \\ + e_5 \left[\left[d^+ \times s^+ \right]^{(2)} \times \left[\tilde{d} \times \tilde{s} \right]^{(2)} \right]_0^{(0)} + e_6 \left[\left[s^+ \times s^+ \right]^{(0)} \times \left[\tilde{s} \times \tilde{s} \right]^{(0)} \right]_0^{(0)} + \\ + e_7 \left[\left[d^+ \times d^+ \right]^{(2)} \times \left[\tilde{d} \times \tilde{s} \right]^{(2)} + \left[d^+ \times s^+ \right]^{(2)} \times \left[\tilde{d} \times \tilde{d} \right]^{(2)} \right]_0^{(0)}\end{aligned} \tag{I-98}$$

where n_d and n_s are the number operators for *d* and *s* bosons.

The connection with the shell model is provided by interpreting the boson as correlated pairs of nucleons [6]

$$S^+ = \sum_j \alpha_j (a_j^+ \times a_j^+)^{(0)} \tag{I-99-a}$$

$$D^+ = \sum_{jj'} \beta_{jj'} (a_j^+ \times a_{j'}^+)_u^{(2)} \tag{I-99-b}$$

Where the summation indices *j*, *j*' run over single particle levels in the valence shell. The connection with the collective model is provided by considering the classical limit of the *s*-*d* boson model.

Introducing a coherent state in this model

$$\left| N, \alpha_u \right\rangle = (s^+ + \sum_u \alpha_u d_u^+)^N \left| 0 \right\rangle \tag{I-100}$$

The classical variables can then be related to the Bohr variables.

I-4-5-Basic Operators

The quadrupole operators is defined in the follows form as [5]

$$Q_u = \left[d^+ \times s\right]^{(2)}{}_u + \left[s^+ \times \tilde{d}\right]^{(2)}{}_u - (\sqrt{7}/2)\left[d^+ \times \tilde{d}\right]^{(2)}{}_u \tag{I-101}$$

with

$$Q^2 = \sum_u (-1)^u Q_u Q_{-u} \tag{I-102}$$

and the pairing operator defined as fallow

$$\hat{P} = \frac{1}{2}\left(\left[\tilde{d} \times \tilde{d}\right]^0 - \tilde{s}\tilde{s}\right) \tag{I-103}$$

Has being considered the linear *U(*6) operators

$$C_1[U(6)] = \sum_{i\ j=1}^{6} G_i^i \tag{I-104}$$

Its eigenvalue is the total number of bosons *N*.

The quadratic Casimir invariant of *U(6)* is given by

$$\begin{aligned} C_2[U(6)] &= \sum_{i=1}^{6} G_i^j G_j^i = \sum b_i^+ b_j b_j^+ b_i \\ &= \sum_{l,l',l'',l'''=0,2} \sum_{K,k} \left[b_l^+ \times \tilde{b}_{l'}\right]_k^{(K)} \left[b_{l''}^+ \times \tilde{b}_{l'''}\right]_{-k}^{(K)} \\ &= \hat{N}(\hat{N}+5) \end{aligned} \tag{I-105}$$

with (l, l' = 0, 2; $b_0 = s$, $b_{2,m} = d_m$)

The linear Casimir invariant of *U*(5) is given by the *d* boson number operator as

$$C_1[U(5)] = \sum_m d_m^+ d_m = \hat{n}_d \tag{I-06}$$

And quadratic Casimir invariant of *U*(5) are computed in the same way in $C_2[U(6)]$

$$C_2[U(5)] = \hat{n}_d(\hat{n}_d + 4) \tag{I-107}$$

The quadratic Casimir operator of SO(6) is given as the follows

$$\begin{aligned}\hat{R}^2 &= C_2[SO(6)] = \frac{1}{2}\sum_{i\,j=1}^{6}\Lambda_{ij}\Lambda_{ji} \\ &= \frac{1}{2}\sum_{ij=1}^{6}(G_i^j - G_j^i)(G_j^i - G_i^j) \\ &= \hat{N}(\hat{N}+4) - \sum_{i=1}^{6} b_i^+ b_i^+ \sum_{j=1}^{6} b_j b_j\end{aligned} \tag{I-108}$$

By using of the coupled notation we find

$$C_2[SO(6)] = \hat{N}(\hat{N}+4) - (d^+\cdot d^+ + s^+ s^+)(\tilde{d}\cdot\tilde{d} + \tilde{s}\tilde{s}) \tag{I-109}$$

The quadratic Casimir invariant of *SO(5)* is computed as

$$\begin{aligned}\hat{T}^2 = C_2[SO(5)] &= -2\sum_{J=1,3}\sqrt{2J+1}\left[\hat{T}^{(J)}\times\hat{T}^{(J)}\right]^{(0)} \\ &= -2\sum_{J=1,3}\sqrt{2J+1}\left[\left[d^+\times\tilde{d}\right]^{(J)}\times\left[d^+\times\tilde{d}\right]^{(J)}\right]^{(0)} \\ &= 2\sum_{J=1,3}\sum_{M}(-1)^M\left[d^+\times\tilde{d}\right]_M^{(J)}\left[d^+\times\tilde{d}\right]_{-M}^{(J)}\end{aligned} \tag{I-110}$$

The Casimir operator of *SU*(3) is found to be

$$C_2[SU(3)] = 2\hat{Q}^2 + \frac{3}{4}\hat{L}^2 \tag{I-111}$$

where

$$\hat{L}_u = \sqrt{10}B_u^{(1)}(2,2) = \sqrt{10}[d^+\times\tilde{d}]_u^{(1)} \tag{I-112}$$

are the generators of the *SO*(3).

The total boson-number operator is given

$$\hat{N} = \sum_k d_k^+ d_k + s^+ s = \hat{n}_d + \hat{n}_s \tag{I-113}$$

The quadratic Casmir invariant of *SO(3)* is given by

$$C_2[SO(3)] = \hat{L}^2 = \sum_\nu (-1)^\nu \hat{L}_\nu \hat{L}_{-\nu} = \sum_\nu (-1)^{(\nu)}[d\times d]_\nu^1[d\times d]_{-\nu}^{(1)} \tag{I-114}$$

The Hamilton operator *H* and the boson number operator, commute $[\hat{H}, N] = 0$.

The Hamilton operator H also commutes with $L_{v;}$ then the total angular momentum Eq. (I-112) is conserved .

$$[\hat{H}\ , L_v]=0 \tag{I-115}$$

One can also express the IBM Hamiltonian (I-98) in terms of the Casmir invariant operators in the form

$$\begin{aligned}\hat{H} = {} & E'_0 + e'_1 \hat{N} + e_2' \hat{N}^2 + e_3' \hat{n}_s + e_4' \hat{n}_d \\ & + e_5' \hat{n}_d^{\ 2} + e_6' \hat{N}\hat{n}_d + e'_7 C_2[SU(3)] + e'_8 C_2[SO(6)] \\ & + e'_9 C_2[SO(5)] + e'_{10} C_2[SO(3)]\end{aligned} \tag{I-116}$$

This Hamiltonian is diagonal in the basis of the group $U(6)$ and its subgroups.

I-4-6.Dynamical symmetry

I-4-4-1.Vibration nuclei: the U(5) limit

In the precedent the Hamilton operator was transformed in a combination of operators which is named Casmir operators .The Hamiltonian characterising the $U(5)$ limit, is obtained by taking $e'_8=e'_7=0$,

$$\begin{aligned}\hat{H}_1 = {} & e'_{12} C_1[U(6)] + e'_{11} C_2[U(6)] + e'_6 C_1[U(6)]C_1[U(5)] + e'_4 C_1[U(5)] \\ & + e' C_2[U(5)] + e'_9 C_2[SO(5)] + e'_{10} C_2[SO(3)]\end{aligned} \tag{I-117}$$

The operator $\hat{H}_1$ can also be written as[6]

$$\hat{H}_1 = a_1 \hat{N} + e'_{11} \hat{N}^2 + (a_2 + e'_6 \hat{N})\hat{n}_d + e'\hat{n}_d^{\ 2} + e'_9 T^2 + e'_{10} J^2 \tag{I-118}$$

The s-boson part corresponds to a simple one-dimensional oscillator, and thus the states may be represented in the form spherical basis

$$\left|[N] n_d \upsilon n_\Delta L M_L\right\rangle = \frac{\left(s^+\right)^{N-n_d}}{\sqrt{(N-n_d)!}} \left| n_d \upsilon n_\Delta L M_L \right\rangle \tag{I-119}$$

where N, n_d ,v , L, and M_L label the irreducible representation of $U(6)$, $U(5)$, $SO(5)$, $SO(3)$, and $SO(2)$, respectively.

The quantum number n_Δ, is the missing label in the $SO(5) \supset SO(3)$ and is related to the maxmun number of *d*-boson triplets coupled to zero angular momentum in the chain $U(5) \supset SO(5) \supset SO(5)$ through the relation

$$n_d = 2n_\pi + 3n_\Delta + \lambda \tag{I-120-a}$$

and

$$\upsilon = 3n_\Delta + \lambda \tag{I-120-b}$$

Then *L* is take the form

$$L = \lambda , \lambda+1 , \ldots , 2\lambda-2 , 2\lambda \tag{I-121}$$

The states $\left|n_d \upsilon n_\Delta L M_L\right\rangle$ are eigenstates of the following operators:

$$\hat{n}_d \left|n_d \upsilon n_\Delta L M_L\right\rangle = n_d \left|n_d \upsilon n_\Delta L M_L\right\rangle \tag{I-122-a}$$

$$C_2[SO(5)]\left|n_d \upsilon n_\Delta L M_L\right\rangle = \upsilon(\upsilon+3)\left|n_d \upsilon n_\Delta L M_L\right\rangle \tag{I-122-b}$$

$$\hat{L}^2 \left|n_d \upsilon n_\Delta L M_L\right\rangle = L(L+1)\left|n_d \upsilon n_\Delta L M_L\right\rangle \tag{I-122-c}$$

$$\hat{L}_0 \left|n_d \upsilon n_\Delta L M_L\right\rangle = M_L \left|n_d \upsilon n_\Delta L M_L\right\rangle \tag{I-122-d}$$

The $U(6) \supset U(5)$ reduction rule takes the form

$$n_d = 0,1, \ldots, N \tag{I-123}$$

while the reduction rule associated whith $U(5) \supset SO(5)$ is given by

$$\upsilon = n_d , n_d-2 , \ldots, 1 \text{ or } 0 \tag{I-124}$$

The eigenvalues for the Hamiltonian $\hat{H}_1$; Eq(I-118) can be represented this way

$$E_1 = e'_{12} N + e'_{11} N(N+5) + e'_6 N n_d + e'_4 n_d + e' n_d (n_d+4) + e'_9 \upsilon(\upsilon+3) + e'_{10} J(J+1) \tag{I-125}$$

This limit corresponds to an anharmonic oscillator in five dimensions, in which there are collective quadruple oscillations about a spherical equilibrium shape.

I-4-4-2.Rotational nuclei: the SU(3) limit

The remaining Hamiltonian consists of the Casimir operators of the second chain (I-95) of algebras as follows [6]

$$\hat{H}_2 = e'_1 \hat{N} + e'_2 \hat{N}^2 + e'_7 C_2[SU(3)] + e'_{10} C_2[SO(3)] \qquad \text{(I-126)}$$

Another form of the Hamiltonian $\hat{H}_2$ can be used

$$\hat{H}_2 = e'_1 \hat{N} + e'_2 \hat{N}^2 + \alpha \hat{J}^2 + \beta \hat{Q}^2 \qquad \text{(I-127)}$$

The $\hat{H}_2$ operator have the eigenstates $|N\ (\lambda,\mu)KLM>$ form. Where N, (λ,μ) , L ,M label the irreducible representation of $U(6)$, $SU(3)$, $O(3)$, $O(2)$, respectively .

We indicate below each algebra the quantum numbers that label their representations [8] .The $SU(3)$ generators is composite of the three angular momentum operators and a quadruple tensor.

The operator $C_2[SU(3)]$ has the following eigenvalues

$$C_2[SU(3)]\ |N\ (\lambda,\mu)KLM> = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda+\mu)\ |N\ (\lambda,\mu)KLM> \qquad , \lambda, \mu \text{ integers} \qquad \text{(I-128)}$$

A detailed treatment of these eigenstates reveals that there exist restrictions for μ and λ. They are positive integers and obey the following rules

$$\mu = 0,2,4,... \qquad \text{(I-129-a)}$$

$$\lambda = 2N - 6l - 2\mu \quad \text{with } l = 0,1, \ldots \text{ and } N = \text{number of bosons} \qquad \text{(I-129-b)}$$

A given doublet (λ,μ) admits only selected values for J. They depend on a ordering number K, for which the following values are permitted

$$K = 0\ ,\ 2,\ 4,\ \ldots\ ,\ \min(\lambda,\mu) \qquad \text{(I-130-a)}$$

For $K = 0$ the values

$$J = 0,2,4,\ \ldots\ ,\ \max(\lambda,\mu) \text{ are allowed} \qquad \text{(I-130-b)}$$

And for $K > 0$

$$J = K, K+1, K+2, \ldots, K + \max(\lambda,\mu) \qquad \text{(I-130-c)}$$

The parameter K is needed for describing the eigenstates. From this analyse one can obtain the eigevalue of the $su(3)$ imit.

$$E_2 = e'_1 N + e'_2 N^2 + e'_{10} J(J+1) + e'_7 (\lambda^2 + u^2 + \lambda\mu + 3(\lambda+\mu)) \quad \text{(I-131)}$$

They don't depend directly on the ordering number K .This limit corresponds to a rigid rotor with axial symmetry

I-4-4-3.γ-instable limit: the SO(6) limit

Finally we come to the dynamical symmetry defined by the group chain (I-96). In this case the $SO(6)$ limit of the IBM corresponds to taking $e'_4=e'_5=e'_6=e'_7=0$ [6].

$$\hat{H}_3 = e'_1 \hat{N} + e'_2 \hat{N}^2 + e'_9 C_2[SO(5)] + e'_{10} C_2[SO(3)] \quad \text{(I-132)}$$

By using an other expression of the Hamiltonian $\hat{H}_3$

$$\hat{H}_3 = e'_1 \hat{N} + e'_2 \hat{N}^2 + \gamma_1 \hat{R}^2 + \gamma_2 \hat{T}^2 + \gamma_3 \hat{J}^2 \quad \text{(I-133)}$$

Choosing e_7' = 0, in Eq((I-132) which means that the Hamiltonian operator is not able any longer to interchange *dd*- with *sd*-states. that there is no interaction between these states.

Since the $SO(6)$ states only differ from the $U(5)$ one in the n_d lable and the latter form a complete basis, it is possible to express the former as

$$\left|[\sigma]\sigma\, \nu\, n_\Delta L M_L\right\rangle = \sum_{n_d} B_{n_d} \left|[\sigma] n_d \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-134)}$$

The quantum numbers N ,σ , ν, L, and M_L are associated with the subalgebras: $U(6)$, $SO(6)$,$SO(5)$, $SO(3)$, and $SO(2)$ respectively . The states (I-134) are eigenstates of the following operators

$$\hat{N}\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle = N\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-135-a)}$$

$$C_2[SO(6)]\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle = \sigma(\sigma+4)\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-135-b)}$$

$$C2[SO(5)]\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle = \nu(\nu+3)\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-135-c)}$$

$$\hat{L}^2\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle = L(L+1)\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-135-d)}$$

$$\hat{L}_0\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle = M_L\left|[N]\sigma\, \nu\, n_\Delta L M_L\right\rangle \quad \text{(I-135-e)}$$

A detailed investigation of the eigenstates of the *SO*(6) algebra reveals that the admissible values for σ are

$$\sigma = N, N - 2, N - 4, ..., 1 \text{ or } 0 \tag{I-136}$$

N is the number of bosons.

Moreover, for the parameter v of the Casimir operator $\hat{T}^2$ (see Eq.(I-110)) the following values are admitted

$$v = 0, 1. . . \sigma. \tag{I-137-a}$$

the number quantum L can amount to

$$L = \lambda, \lambda + 1. . . 2\lambda - 3, 2\lambda - 2, 2\lambda \tag{I-137-b}$$

with

$$\lambda = v - 3n_\Delta \tag{I-137-c}$$

and

$$n_\Delta = 0, 1, \ldots \tag{I-137-d}$$

The energy eigenvalue of the *SO(6)* limit ,is given

$$E_3 = e'_1 N + e'_2 N^2 + \gamma_1\sigma(\sigma+4) + \gamma_2 v(v+3) + \gamma_3 J(J+1) \tag{I-138}$$

The 2^+- 4^+-doublet of the third and fourth level (with v = 2) is characteristic for this special case and appears also in the so called γ-instable nuclei, for which reason this model is named γ -instable limit as well.

CHAPTER II :

Microscopic cluster model

II-1. The clustering phenomenon

The formation of clusters is a fundamental aspect of nuclear many body dynamics together with the formation of mean field. Clustering aspects appear abundantly in many problems in both nuclear structure and nuclear collisions. The coexistence of the clustering aspect and the single particle aspect is a unique feature of nuclear many body systems.

The basic assumption of this model is such that nuclei can be described accurately in terms of a system of two-component nuclei; each with its free state characteristics; interacting through a deep local potential. However, many binary decompositions satisfy this minimal requirement. Recently, Buck et al. proposed that the choice must be done with reference to the binding energies of the cluster and core.

In light stable nuclei it is well known that the clustering structure is of basic importance .In the region of very light stable nuclei with A less than about 10, the clustering structure shows up in ground states. In heavier mass region of stable nuclei, however, the mean field is formed in ground states and the clustering structure appears in excited states.

States in nuclei that is have α-particles clusterizations with $N = Z$ are typically not found in ground states, but are observed as excited states close to the decay thresholds into clusters, as was suggested in 1968 by Ikeda. The The Ikeda diagram is shown in Fig. 1, this links the energy required to liberate the cluster constituents to the excitation energy at which the cluster structures prevail in the host nucleus. The clear prediction, is that cluster structures are mainly found close to cluster decay thresholds.

clustering gives rise to states in light nuclei which are not reproduced by the shell model, but the nuclear shell model does, however, play an important role in the emergence of nuclear clusters, and also in the description of special deformed nuclear shapes, which are stabilised by the quantal effects of the many-body system, namely the deformed shell gaps.

This connection is illustrated by the behaviour of the energy levels in the deformed harmonic oscillator], shown in Fig. 2. The numbers in the circles correspond to the number of nucleons,

which can be placed into the crossing points of orbits. At zero deformation there is the familiar sequence of magic numbers which would be associated with spherical shell closures, and the associated degeneracies. At a deformation of the potential, where the ratios of the axes are 2:1, these same magic degeneracies reappear, but are repeated twice. This establishes an explicit link between deformed shell closures and clustering. This concept, fundamental for the understanding of the appearance of clustering within the nucleus,. the deformed magic structures with special stability are expected for particular combinations of spherical (shell-model) clusters. For example(see table I), for super-deformed structures (2:1) the magic numbers have a decomposition into two magic numbers, of two spherical clusters, e.g. ^{20}Ne≡ *(*^{16}O+α). Thus, one would expect clusterisation not only to appear at a particular excitation energy, but also at a specific deformation. The Hyperdeformation (3:1) are related to cluster structures consisting of three clusters. For larger deformations longer *α*-chain states are produced.

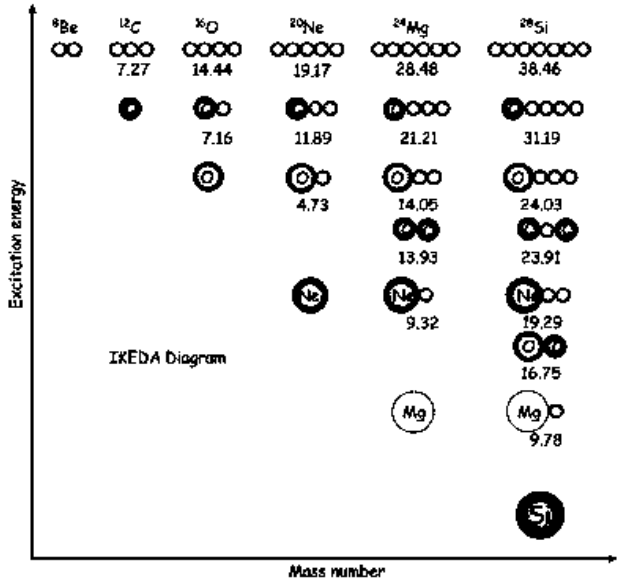


Fig. 1. The Ikeda threshold diagram for nuclei with *α*-clustering. Cluster structures are predicted to appear close to the associated decay thresholds.These energies needed for the decomposition of the normal nucleus into the structures are indicated in MeV.

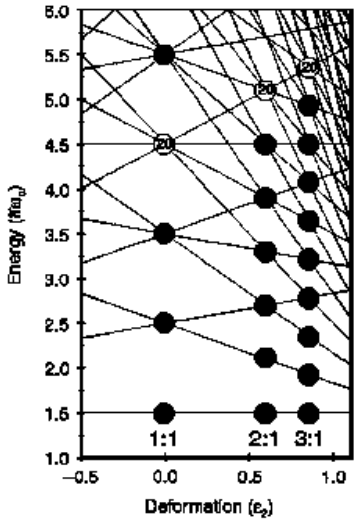


Fig. 2. Energy levels of the deformed axially symmetric harmonic oscillator as a function of the quadrupole deformation (oblate and prolate, i.e.negative and positive values of ε_2, respectively). Degeneracies appear due to crossings of orbits at certain ratios of the length of the long axis (the symmetry axis) to the short perpendicular axis. The regions of high degeneracy define a shell closure also for deformed shapes.

Deformed	spherical constituents	
N	Superdeformation, dimers	
4	2+ 2	α–α
10	8 + 2	^{16}O–α
16	8 + 8	^{16}O–^{16}O
28	8 + 20	^{16}O–^{40}Ca
N	Hyperdeformation, chains	
6	2+ 2 + 2	α–α–α
12	2 + 8 + 2	α–^{16}O–α
24	8 + 8 + 8	^{16}O–^{16}O–^{16}O
36	8 + 20 + 8	^{16}O–^{40}Ca–^{16}O
48	20 + 8 + 20	^{40}Ca–^{16}O–^{40}Ca
60	20 + 20 + 20	^{40}Ca–^{40}Ca–^{40}Ca
N	Oblate nuclei, pancakes	
8	6+2	^{12}C–α
12	6+6	^{12}C–^{12}C
18	12+6	^{24}Mg–^{12}C
24	12+12	^{24}Mg–^{24}Mg

Table 1-The constituents for the nucleon magic numbers *N* in nuclei with super- and hyper-deformed prolate shapes

II-2.Basic idea

The simplest possible binary cluster model has been shown to account well for many of the observed features of various light nuclei. The basic assumption of this model is that such nuclei can be described accurately in terms of a system of two component nuclei. We initially consider a nucleus. (Z_T ;A_T) decomposed into the single core-cluster configuration (Z_1;A_1)+(Z_2;A_2), each with its free state characteristics; interacting through a deep local potential. However, much binary decomposition satisfies this minimal requirement. This choice must be done with reference to the binding energies of the cluster and core. if the parent nucleus can divide into a cluster and core which are both doubly magic, then this will be the most favoured combination.

In the present case , where one or both of the subnuclie may be excited , we have to solve a set of coupled Schrödinger equation to find the desired wave functions and corresponding energy eigenvalues. Consider the tow-body reaction described by the Hamiltonian

$$H=T+H_1(r_1)+H_2(r_2)+V(r_1,r_2) \qquad \text{(II-1)}$$

where T is he kinetic energy operator for he relative motion , V is the interaction between the tow subnucleus, and H_1 and H_2 are the subnucleus Hamiltonian . Thus ,for cluster we have

$$H_1(r_1)\Phi_{In\,Kn}(r_1) = \varepsilon_{na}\,\Phi_{In\,Kn}(r_1) \qquad \text{(II-2)}$$

Where $\Phi_{In\,Kn}(r_1)$ is the internal wave function for the cluster in the nth state, of energy ε_{na}, having spin and projection I_n and K_n, respectively. We wish to find the bound states of the Hamiltonian of Eq (II-1) .

If the total wave function of the system is $\psi_{JM}(r,R)$,the time independent Schrödinger equation is

$$H\,\psi_{JM}(r,R)=E_J\,\psi_{JM}(r,R) \qquad \text{(II-3)}$$

where E_J is the total energy of the system in he state $\psi_{JM}(r,R)$ .This total wave function may be expanded in terms of radial functions and harmonic spherical

$$\psi_{JM}(r) = \sum_{n,l,m} \chi_{nL}(r) y_{lm}(\theta,\varphi) \tag{II-4}$$

If we substitute the expansion of Eq (II-3) into the Schrödinger equation .The energies and wave functions of relative motion are obtained by solving the radial Schrödinger [10]

$$\frac{-\hbar^2}{2\mu}\frac{d^2\chi_{nL}}{dr^2} + \left[\frac{\hbar^2 L(L+1)}{2\mu r^2} + V_N(r) + V_C(r)\right]\chi_{nL}(r) = E_{nL}\chi_{nL}(r) \tag{II-5}$$

Cluster model description of a given nucleus is the identity of the cluster and core to be used, that suggested for a nuclear molecule-like structure to appear, its excitation energy needs to be near or above the threshold energy for breakup into the constituent clusters (and also below the top of the potential barrier).

II-3.The appropriate core–cluster decomposition

The choice of the appropriate core–cluster decomposition of a given nucleus is clearly of fundamental importance when applying a cluster model to that nucleus.

We assume that the preferred modes of binary clusterization are associated with the greatest stabilities of core and cluster . Out of all possible partitions of a nucleus of total charge and mass $(Z_T ; A_T)$ into a core $(Z_1;A_1)$ and a cluster $(Z_2;A_2)$ we therefore seek the maxima of the quantities[9]

$$D(1,2) = \left[B_A(Z_1,A_1) - B_M(Z_1,A_1)\right] + \left[BA(Z_2,A_2) - B_M(Z_2,A_2)\right] \tag{II-6}$$

Where, B_A and B_M are the actual binding energy and the corresponding liquid drop value, respectively.

The value of B_M is calculated using the Weiszacker formula

$$B_M = a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta \tag{II-7}$$

where

a_V =15.56 MeV, a_S =17.23 MeV, a_C =0.7 MeV, a_a =23.285 MeV and $\delta = 12/A^{1/2}$Mev (II-8)

For a given nucleus, when the conditions $A_1 = A_T - A_2$ and $Z_1 = Z_T - Z_2$ are applied, *D(1,2)* remains a function of two independent variables, the cluster mass and charge, (A_2,Z_2). A simpler form of *D(1,2)* resulting from the observation that electric dipole transitions between low-laying bands of opposite parity in heavy nuclei are very weak.

We restrict ourselves to decompositions of even–even nuclei into even–even fragments. A further constraint follows from the observed smallness of *B(*E1*)/B(*E2*)* ratios in heavy nuclei. This requires that

$$\frac{Z_1}{A_1} = \frac{Z_2}{A_2} = \frac{Z_T}{A_T} \tag{II-9}$$

In general equation (II-9) leads to non-integral values of (A_1,Z_1) and (A_2,Z_2) which can be interpreted as averages arising from suitably weighted mixtures of cores and clusters . *D* then remains a function of one variable, the average cluster charge $<Z_2>$.

In the present calculations, for each cluster charge Z_2 the masses, A_2 and $A_2 + 2$, which come closest to satisfy this no-dipole constraint, are such that [10]

$$\frac{Z_2}{A_2} \geq \frac{Z_T}{A_T} \geq \frac{Z_2}{A_2 + 2} \tag{II-10}$$

The maximum stability criterion has applied successfully to parts of the periodic table where the choice of core and cluster is by no means obvious. However, for many mid-shell nuclei, this method yields poor results, possibly because of shape changes of core or cluster away from the spherical [11]. It is thus important to determine whether it is the cluster model, which is breaking down for these nuclei, or merely the maximum stability method of generating the core–cluster decompositions.

II-4.Cluster-core potential

The form adopted for the cluster-core potential *V(r)* [12]

$$V_{\text{int}}(r,R) = V_N(r,R) + V_C(r,R) + \eta^2 \frac{(l+1/2)^2}{2\mu r^2} \tag{II-11}$$

Where, V_N is the nuclear modified Woods–Saxon potential

$$V_N(r,R) = -\frac{A_1 A_2}{A} V_0 \frac{F(r,R,x,a)}{F(0,R,x,a)} \tag{II-12}$$

V_0, x and a is parameter values indicated in section (II-9-1-1)

where

$$F(r,R,x,a) = \frac{x}{1+\exp\left[(r-R)/a\right]} + \frac{1-x}{1+\exp\left[(r-R)/(3a)\right]^3} \tag{II-13}$$

Analyses of an elastic scattering have shown that optical potentials using a Saxon-Woods (SW) shape are unable to reproduce details of the experimental cross sections. For light nuclei, variants based on a $(SW)^2$ shape have been found useful in studies of bound and quasi-bound α-cluster states, as well as of low energy an elastic scattering. In similar analyses involving nuclei throughout the periodic table, several authors have found it better to employ the mixed (SW) + $(SW)^3$ form [14]

$$\begin{aligned} V(r) &= -V_0 \left[\frac{x}{1+\exp\left[(r-R)/a\right]} + \frac{1-x}{1+\exp\left[(r-R)/(3a)\right]^3} \right] \\ &= \mathrm{V}_0 f(r,x,a,R) \end{aligned} \tag{II-14}$$

Moreover, V_C is the general Coulomb potential taken to be that acting between a uniformly charged spherical core of radius R and a point cluster, completely defines the core–cluster interaction [13].

$$V_C = C/r \qquad r \geq R \tag{II-15-a}$$

$$V_C = C\frac{3-(r/R)^2}{2R} \qquad r \leq R \tag{II-15-b}$$

II-5. Hamiltonian diagonalisation

In order to solve the Schrödinger equation numerically a system of basis stats $\{|u_i\rangle\}$ are needed. We have chosen eigenstats of the isotopic harmonic oscillator $|Nlm\rangle$ (see Appendix A). H is then represented by a matrix whose elements are given by

$$H_{ij} = \langle u_i | H | u_j \rangle \tag{II-16}$$

where

$$H_{ij} = \int \chi_{N_i l_i}(r) Y_{l_i m_i}(\theta,\varphi) H \chi_{N_j l_j}(r) Y_{l_j m_j}(\theta,\varphi) r^2 dr d\Omega \tag{II-17}$$

and

$$\Psi = \sum_{Nlm} C_{Nlm} \chi_{Nl}(r) Y_{Nl}(\Omega) \tag{II-18}$$

The radial function $\chi_{N_i l_i}(r)$ are given by (I-9) and *N=2(n-1)+l*.

If H has spherical symmetry then

$$\begin{aligned} H_{ij} &= \int \chi_{N_i l_i}(r) Y_{l_i m_i}^{*}(\theta,\varphi)\left[\frac{-\hbar^2}{2\mu}\frac{d^2\chi_{nL}}{dr^2} + \left[\frac{L^{\to 2}}{2\mu r^2} + V_N(r) + V_C(r)\right]\right]\chi_{N_j l_j}(r) Y_{l_j m_j}(\theta,\varphi) r^2 dr d\Omega \\ &= \int \chi_{N_i l_i}(r)\left[\frac{-\hbar^2}{2\mu}\frac{d^2\chi_{N_j l_j}}{dr^2} + \left[\frac{\hbar^2 l_j(l_j+1)}{2\mu r^2} + V_N(r) + V_C(r)\right]\right]\chi_{N_j l_j}(r) r^2 dr \quad \int Y_{l_i m_i}^{*}(\theta,\varphi) Y_{l_j m_j}(\theta,\varphi) d\Omega \end{aligned} \tag{II-19}$$

From the orthogonality of the spherical harmonics

$$\int Y_{l_i m_i}^{*}(\theta,\varphi) Y_{l_j m_j}(\theta,\varphi) d\Omega = \delta_{l_i l_j} \delta_{m_i m_j} \qquad \text{(II-20-a)}$$

one gets

$$l_i = l_j = L,, \;\; m_i = m_j \qquad \text{(II-20-b)}$$

H_{ij} reduce then to (II-21)

$$H_{ij} = \int \chi_{N_i l}(r) \left[\frac{-\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2 L(L+1)}{2\mu r^2} + V_N(r) + V_C(r) \right] \chi_{N_j l}(r) r^2 dr$$

and

$$\Psi_{KLm} = \sum_{Nlm} C_{Nl} \chi_{Nl}(r) Y_{Lm}(\Omega) \qquad \text{(II-22)}$$

The harmonic oscillator basis is infinite, a truncation is then necessary. For a given L, H_{ij} is calculated. The upper value of N is related to convergence of calculation.

II-6. Quantum numbers and Wildermuth condition

For simplicity we initially consider a nucleus (Z_T, A_T) decomposed into the single core cluster configuration $.(Z_1,A_1)+(Z_2,A_2)$. Possible states of this system group into bands, with each band labelled by its value of the global quantum number $G = 2N+L$, where N is the number of nodes and L the angular momentum of a state in the band. An even value of G thus corresponds to a band of positive parity states having

$$L^{\pi} = 0^{+}, 2^{+}, 4^{+}, \ldots, G^{+}. \qquad \text{(II-23)}$$

The semi-classical quantization condition for the head of a band of states characterized by a fixed value of G is [12]

$$\int_{r1}^{r2} dr \sqrt{\frac{2\mu}{\hbar^2}\left[E - V_{\text{int}}(r,R)\right]} = (G - L + 1)\frac{\pi}{2} \qquad \text{(II-24)}$$

where, r_1 and r_2 are the two inner most classical turning points and μ is the relative mass of the core–cluster decomposition. *V* is the interacting potential between the cluster and the core that depends on the relative distance *r* and the core radius *R*.

The choice of the relative motion *G*-value is guided by the Wildermuth condition and the spherical shell model. If we were describing the cluster and core nucleon orbitals by harmonic oscillator wave functions, with a common length parameter, the minimum possible value of G could be obtained by requiring all the cluster nucleons to occupy states above the Fermi surface of the core nucleus.

For a ^{208}Pb core, each proton orbital would have $G_p=2n_p+l_p=5$, and each neutron orbital $G_n=2n_n+l_n=6$. A ground state ^{24}Ne nucleus consisting of $(0s)^4$ $(0p)^{12}$ $(1s0d)^8$ nucleons requires 28 quanta for its construction. Summing the individual cluster nucleon contributions, and subtracting $G_{\rm int}$ associated with the shell model ground state structure of the cluster, would then lead to a well-defined value of $G= G_pZ_c+ G_n N_c-G_{\rm int}$ = (10 × 5) + (14 × 6) - 28 = 106. Here *Zc* and *Nc* are the numbers of protons and neutrons of the cluster, respectively.

A widely used approach is to choose *G* as the lowest value compatible with the Wildermuth condition, for light nuclei with N < 82 using, for example, G = 4, 8, 12 and 16 for the ground state bands of the α plus closed core nuclei ^{8}Be, ^{20}Ne, ^{44}Ti and ^{94}Mo, respectively. For heavier nuclei has been found that good fits to the α-decay data require the change ΔG = 2 expected on crossing the N = 126 shell closure, but that the effects of the Z = 82 shell closure appear to be weak, requiring no associated change in G. then ,one should use G = G ≤18 for 82 < N < 126, and G = G≥ 20 for N > 126.

For the lowest of such bands in the $A_T \approx$ 150 mass region have used $G = 4A_2$ and where A_2 is the mass number of the cluster. This prescription for G for the ground state band is compatible with the treatments of the actinide nuclei, where the cluster nucleons were all outside the Z = 82, N = 126 shells .For $A_T \approx$ 230 and has been taken[15] G = $5A_2$. Here the cluster nucleons are outside the Z = 50, N = 82 shells, and so has been reduce the G-value by one unit per cluster nucleon.

For superdeformed cluster, the systematics suggest that G values for different binary clusterizations in the same nucleus are approximately proportional to A_1A_2. Taking values of G of ground state band in ^{194}Hg, ^{236}U and ^{240}Pu to be in the interval 14–18, 116–120 and 126–140, respectively, thus has expected the values of G in superdeformed clusterizations to be in the ranges 140–180, 312–322 and 302– 336, respectively.

The G values for different binary decompositions of different nuclei in similar considerations is obtained here as

$$G\alpha \frac{RA_1A_2}{(A_1+A_2)} \approx \frac{0.88A_1A_2}{(A_1+A_1)^{2/3}} \qquad \text{(II-25)}$$

Where R is potential radius.

This indicates that rounding Eq. (II-25) to the nearest even integer gives a fair approximation to G for positive parity bands in even–even nuclei. From this consequence thus, it is possible to obviate to treat G as an adjustable parameter.

II-7. Electromagnetic Transition

The energies of the members of the ground state band in each of the heavy nuclei are obtained directly by solving the Schrödinger equation, with the potential of the section 4, whose energy separations are less than those expected from a $J(J + 1)$ spectrum having a constant moment of inertia determined from the excitation energy of the 2^+ state.

In addition to eigenenergies, the solution of the Schrödinger equation yields radial wave functions that may be used to calculate electromagnetic transition rates. Reduced transition strengths within the ground state band are calculated in a binary cluster model

In these applications an even-even nucleus of charge Z and mass A is taken as consisting of a spinless core (Z_1 ,A_1) and spinless cluster (Z_2 ,A_2), interacting via a central two-body interaction $V(r)$.

The reduced matrix elements of the electric transitions of multipolarity λ is [13]

$$\left[ME(\lambda)\right]_{J_i J_f} =< J_f \left\| ME(\lambda) \right\| J_i >= f(J_i, J_f, \lambda)\alpha_\lambda < r^\lambda > \tag{II-26}$$

with

$$f(J_i, J_f, \lambda) = \sqrt{\frac{(2j_i+1)(2j_f+1)}{4\pi}} \left\langle J_i 0 J_f 0 \middle| \lambda 0 \right\rangle \tag{II-27}$$

and

$$\alpha_\lambda = \frac{Z_1 A^\lambda{}_2 + (-1)^\lambda Z_2 A^\lambda{}_1}{(A_1 + Z_1)^\lambda} \tag{II-28}$$

Zi and *Ai* are the charge and mass of cluster and core, respectively, and an effective charge ϵ can be introduced by substituting $Zi \rightarrow Zi + \epsilon Ai$, which corresponds to giving neutrons a charge of ϵ *e* and increasing the proton charge to (1+ϵ)*e*.

The relation between $\left\langle J_f \left\| M(E2) \right\| J_i \right\rangle$ and *B*(*E*2) is defined from the Wigner-Eckart theorem

$$\left\langle J_f \left\| M(E2) \right\| J_i \right\rangle = \sqrt{B(E2\downarrow)(2J_i+1)} \tag{II-29}$$

We have

$$B(E_\lambda \uparrow; 0^+ \rightarrow \lambda) = \frac{2\lambda+1}{4\pi}\alpha_\lambda^2 < r^\lambda >^2 \ \mathrm{e}^2\mathrm{fm}^{2\lambda} \tag{II-30}$$

We remove the sign and explicit *J* dependence of the matrix elements by defining

$$\left[ME(\lambda)\right]_{J_i J_f}{}' = \left| \left[ME(\lambda)\right]_{J_i J_f} / f(J_i, J_f, \lambda) \right| \tag{II-31}$$

In this model has being considered both cluster and core are spinless entities, so the magnetic dipole moment of an excited state of angular momentum J is entirely due to the relative orbital motion of the two bodies , and takes the simple relation

$$< \mu >_J = \left[\frac{(A_1^2 Z_2 + A_2^2 Z_1)}{A_1 A_2 (A_1 + A_2)} \right] \tag{II-32}$$

where Z_1 and A_1 are the charges and masses of cluster and core.

Using the radial wave functions obtained from solving the Schrödinger equation to calculate reduced *E2* transition strengths between states *J* and *J - 2* according to the Formula

$$\begin{aligned} B(E2; J_i \to J_f) &= \frac{\alpha_2^2}{4\pi}(2J_f + 1)\left\langle J_i 0 J_f 0 \middle| 20 \right\rangle^2 < r^2 >^2 \\ &= \frac{15\alpha_2^2}{8\pi}\frac{L(L-1)}{(2L+1)(2L-1)} < r^2 >^2 \end{aligned} \tag{II-33}$$

Where $<r^2>$ is the integral of the squared cluster-core separation distance *r*, multiplied by initial and final state radial wave functions, where

$$\alpha_2 = \frac{Z_1 A_2^{\,2} + Z_2 A_1^{\,2}}{(A_1 + A_2)^2} \tag{II-34}$$

In some cases extensive *E*2 data are available from Coulomb excitation measurements, and are presented as the reduced matrix elements $<J_f \| M(E2) \| J_i>$ rather than as *B*(*E*2) strengths.

One consider two such states of angular momenta , and *L*, energies E_l and E_L , and radial wave functions χ and χ_L [16] , respectively

$$\frac{2\mu}{\hbar^2}(E_L - E_l)\int_0^\infty r^2 \chi_L \chi_l dr = \left[L(L+1) - l(l+1)\right]\int_0^\infty \chi_L \chi_l dr - \int_0^\infty 2r\left[\chi_L \frac{d\chi_l}{dr} - \chi_l \frac{d\chi_L}{dr}\right]dr \tag{II-35}$$

The angular momentum $L^\pi = 0^+, 2^+, 4^+, \ldots$.

For the low angular momentum multimodal states of interest here, the cluster model produces closely similar surface peaked radial wave functions.

By replacing the first integral on the right hand side of Eq. (II-35) by unity, and neglecting the second, we obtain

$$\mu\int_0^\infty r^2\chi_L\chi_l dr \approx \frac{\hbar^2}{2}\left\{\frac{L(L+1)-l(l+1)}{E_L-E_l)}\right\} \tag{II-36}$$

so that

$$\mu=\frac{\mathrm{A}_1\ \mathrm{A}_2}{\mathrm{A}}=\frac{\mathrm{Z}_1\ \mathrm{Z}_2}{\mathrm{Z}}\ \frac{\mathrm{A}}{\mathrm{Z}} \tag{II-37}$$

The electric quadrupole transition strength is

$$\begin{aligned} BE2 &= \frac{1}{4\pi}\left[\left(\frac{Z_1A^2{}_2+Z_2A^2{}_1}{(A_1+Z_1)^2}\right)\int_0^\infty r^2\chi_2\chi_0 dr\right]^2 \\ &\approx \frac{1}{4\pi}\left[\frac{Z_1Z_2}{Z}\int_0^\infty r^2\chi_2\chi_0 dr\right]^2 \\ &\approx \frac{1}{4\pi}\left[\frac{Z_1Z_2}{Z}\int_0^\infty r^2\chi_L\chi_l dr\right]^2 \end{aligned} \tag{II-38}$$

Supposed closely similar radial wave functions.

$$BE2 \approx \frac{1}{4\pi}\left[\frac{\hbar^2}{2}\left\{\frac{L(L+1)-l(l+1)}{E_L-E_l)}\right\}\frac{Z}{A}\right]^2 e^2\,fm^4 \tag{II-39}$$

We note that Eq .(II-39) does not depend on the specific form of $V(r)$, as long as the potential is common to all members of the band.

Recently has being proposed a relation between the B ($E2$: $2^+ \rightarrow 0^+$) value of a nucleus (A, Z) and the reduced mass μ of its core–cluster relative motion. Specifically

$$\mu = \frac{A_1A_2}{A} = \frac{\sqrt{4\pi B(E2)}A^{1/3}}{r_0^2 Z} \tag{II-40}$$

Which, together with the no-dipole constraint of equation (II-10) and a value for r_0 enables us to generate the required core (A_1,Z_1) and cluster (A_2,Z_2).

II-8.Application to Er isotopes

II-8-1.Energy levels

We choose a cluster charge Z_2, then the masses A_2 and A_2 +2, which come closest to satisfying the no-dipole constraint (see Eq(II-10)).

We thus restrict our attention to two cluster masses for each cluster charge. Fig. (3-1,3-2,3-3,3-4,3-5,3-6,3-7) shows the values of $D(Z_I, A_1, Z_2, A_2)$ for the lower and higher of these two masses .From thesis figures one can notice that there are two maxima at $Z_2 = 2$ and $Z_2 = 6$ in $158 \leq A < 164$ region for these two masses, in the case A=164 we have three maxima at $Z_2=2$,$Z_2=6$, and $Z_2=10$ for the higher of these tow masses and we have two maxima at $Z_2=2$ and $Z_2=10$ for the lower of these tow masses. while in the case A=166 we have two maxima at $Z_2=2$ and $Z_2=12$ for the lower of these tow masses and we have two maxima at $Z_2=2$ and $Z_2=10$ for the higher of these tow masses.

In ^{168} Er and^{170} Er istopes region we have two maxima at $Z_2=2$ and $Z_2=12$ for the higher of these tow masses. In this work we considered for the lower cluster masses show that ^{12}C ,^{14}C and^{14}C might be a good cluster for ^{158}Er ^{160}Er and ^{162}Er ,but in cluster higher masses show that ^{26}Ne ,^{26}Ne ,^{30}Mg and ^{32}Mg is considered as a cluster nucleus in ^{164}Er ^{166} Er ^{168} Er and ^{170} Er.

The studies in the rare-earth and actinide regions have shown that $G = 4A_2$. This is relation has been used in our calculations (see table 2).

The value of the potential radius R, is determined from the Bohr–Sommerfeld Eq(II-24) relation by fitting the energy of a single level of the band, usually the lowest state. We can then settle on that combination of G and R that gives energy closest to the experimental value for the highest known band member.

II-8-1-1.Method of calculation

In this model both cluster and core nuclei are spinless. The energies of the members of the ground state band in each of the Er isotopes under consideration are obtained directly by solving the Schrödinger equation for the required values of the L using Numerical method, by projected the wave function on harmonic oscillator wave functions basics .This achieved using Jacobean subroutine (see Appendix B) ,in this method the minimum number N required for convergence

of the calculation is 20 . The frequency parameter is taken as a free parameter and its value is adjusted through the χ^2-test .This is a straightforward improvement cluster model calculations in which frequency parameter is determined by fitting the energy of a single level of the band, usually the lowest state with. For the nuclear interaction $V_N(r,R)$ we use a modified Woods–Saxon potential (see Eq. (II-17))with standard parameter fixed to the following values V_0 = 54.0 MeV, a = 0.73 fm and x = 0.33. This, together with a Coulomb potential $V_C(r,R)$ defined by Eq.(II-15),($c=z_1.z_2$ $2.3077213.10^{-28}.j.m$).

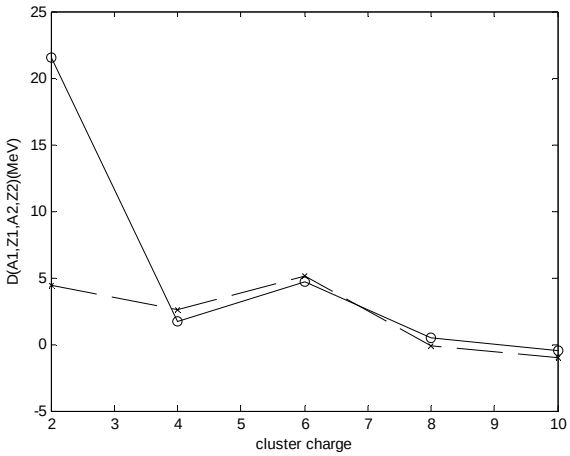


Figure 3-1. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{158}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

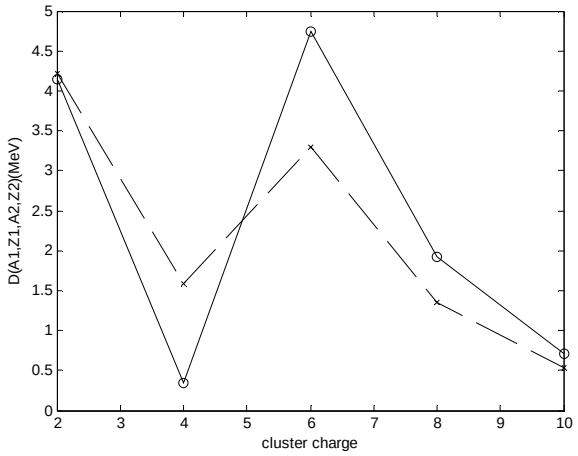


Figure 3-2. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{160}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

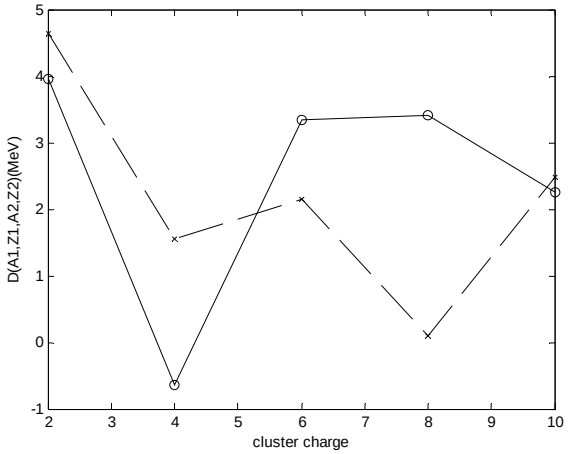


Figure 3-3. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{162}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

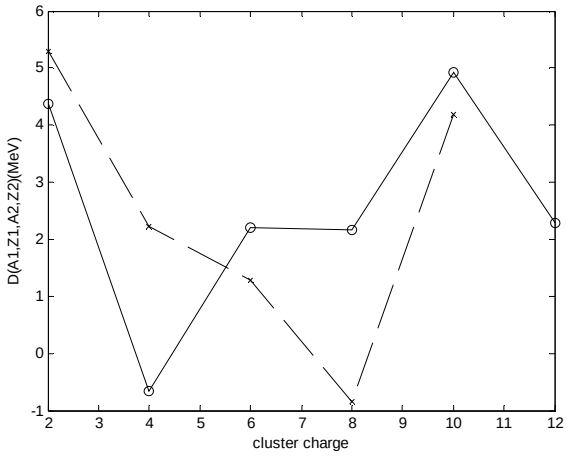


Figure 3-4. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{164}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

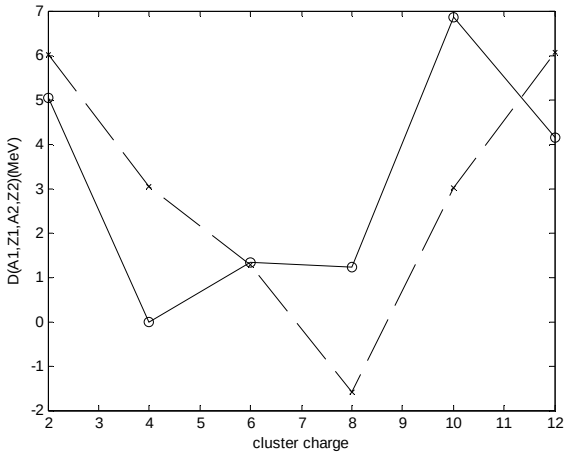


Figure 3-5. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{166}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

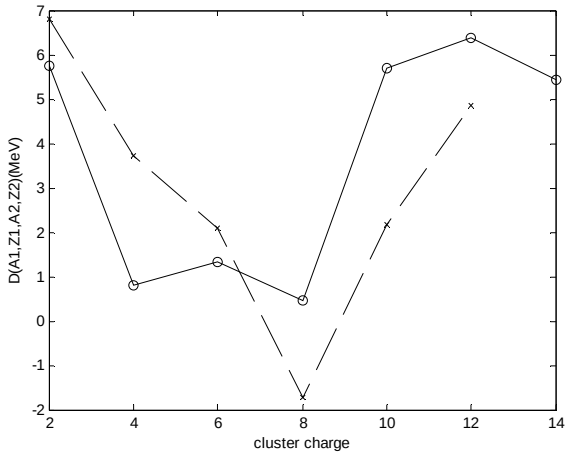


Figure 3-6. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{168}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

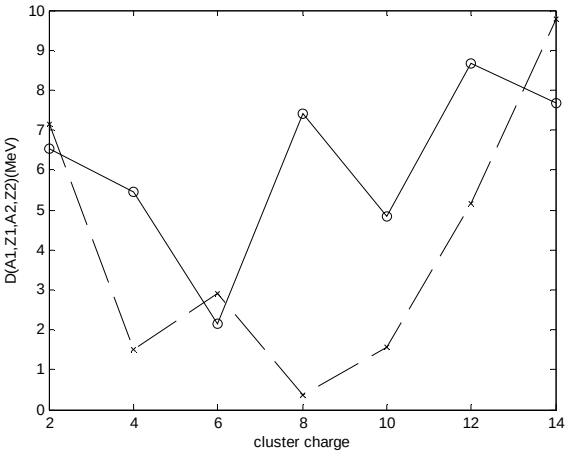


Figure 3-7. Calculations of $D(A_1,Z_1,A_2,Z_2)$ as a function of cluster charge for ^{170}Er. dashed line represent cluster with lower mass number and solid line represent cluster with higher mass number .

tendency for the spectra of a sequence of isotopes to become more compressed as the mass number increases.

For Er isotope the available information on energy levels only goes up to 8^+ in the ground state

Apart from the usual difficulties in describing the 0^+-2^+ spacings for ^{158}Er, the lower reaches of the spectra are well reproduced up to spins of about 10^+ for the ^{158}Er and about 12^+ for ^{160,162,164,166}Er. This problem gets progressively worse for higher spins, and the values for the 14^+ states in Er isotopes are about 0.4 MeV below the measured values. Thus The calculated energies of the states in the ^{158}Er are now slightly overestimated, but those of ^{160,162,164,166}Er are much improved, so much so, that the spectrum for ^{164}Er represents one of the best fits we have been able to achieve. For $^{166\text{-}168\text{-}170}$Er, we find a best fits for 0^+-2^+-4^+- 6^+, spacings with measurement values .

We find it remarkable that the properties ground state bands can be well reproduced by a binary cluster model with reasonable values for the two free parameters.

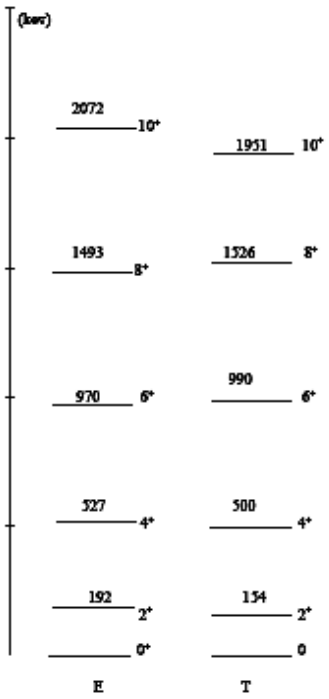


Figure 4-1. Ground state band excitation energies for ^{158}Er calculated (T) using a ^{12}C cluster,R = 5.6988 fm, compared to experiment (E).

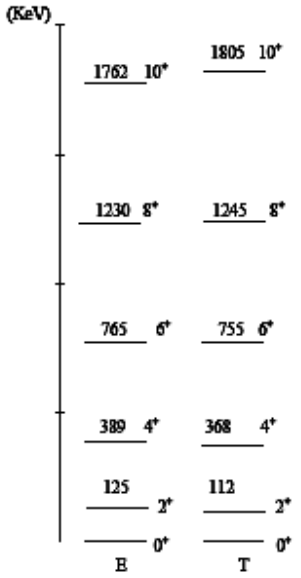


Figure 4-2. Ground state band excitation energies for ^{160}Er calculated (T) using a ^{14}C cluster,R = 5.8878 fm, compared to experiment (E).

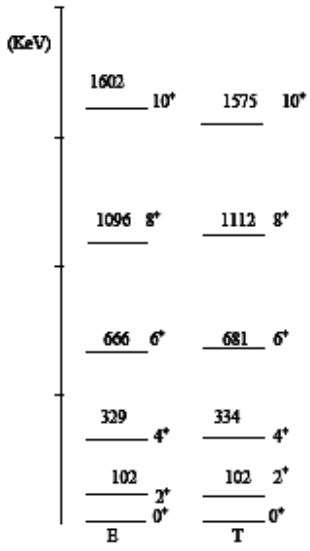


Figure 4-3. Ground state band excitation energies for ^{162}Er calculated (T) using a ^{14}C cluster,R = 5.8878 fm, compared to experiment (E).

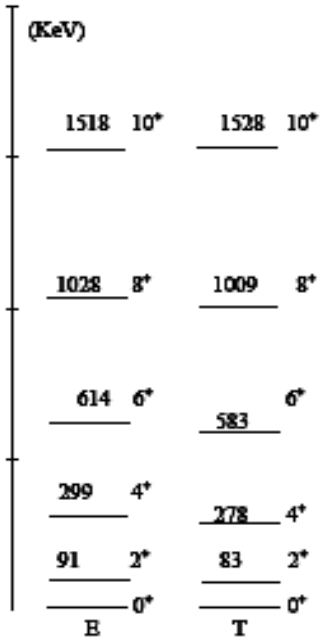


Figure 4-4. Ground state band excitation energies for ^{164}Er calculated (T) using a ^{14}C cluster,R = 5.8824 fm, compared to experiment (E).

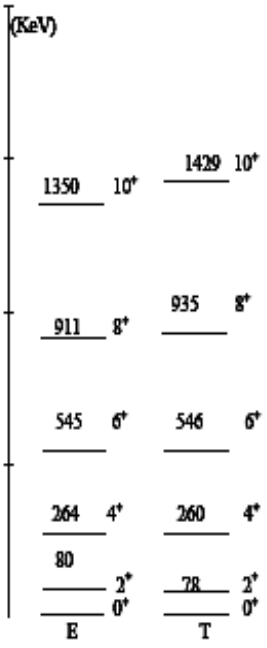


Figure 4-5. Ground state band excitation energies for ^{166}Er calculated (T) using a ^{26}Ne cluster,R = 7.7346 fm, compared to experiment (E).

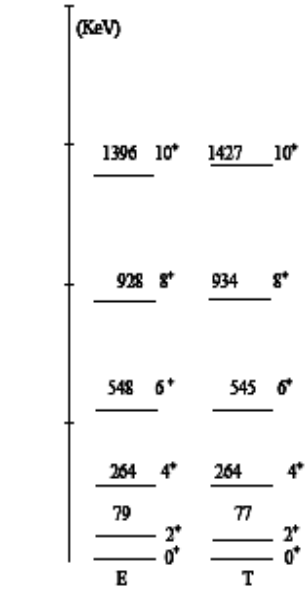


Figure 4-6. Ground state band excitation energies for ^{168}Er calculated (T) using a ^{26}Ne cluster,R = 7.7238 fm, compared to experiment (E).

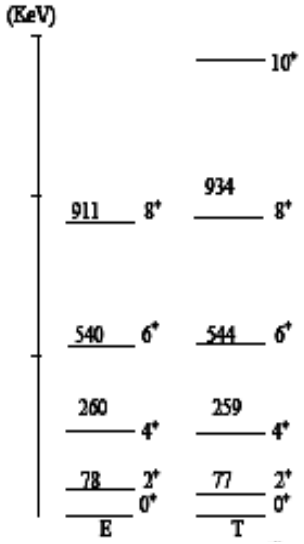


Figure 4-7. Ground state band excitation energies for ^{170}Er calculated (T) using a ^{30}Mg cluster,R = 8.3232 fm, compared to experiment (E)

II-8-2.Electric quadrupole transitions

The value of *B*(*E*2) acts as a good indicator of the dominant cluster to be employed in our model .The transition probability *B(E2)* of a nucleus (A,Z) is given in terms of the reduced mass μ of its core–cluster relative motion by following relation

$$B(E2) = \frac{\mu^2 Z^2 r_0^4}{4\mu\pi A^{2/3}} \tag{II-41}$$

Where r_0 can be assumed as a free parameter.

The experimental $B(E2:2_1^+ \rightarrow 0_1^+)$ (in e^2b^2) can be obtained from the measured half-live of the 2_1^+ stats as:

$$B(E2 = 2_1^+ \rightarrow 0_1^+) = \frac{1}{1.24\times10^{13}} \frac{1}{E^5} \frac{\ln(2)}{T_{1/2}} \frac{1}{1+\alpha_T} \tag{II-42}$$

Where E is the 2_1^+ Energy in MeV and α_T is the total internal conversion factor.

The value of r_0 needed to obtain agreement with experimental *B*(*E*2) transition strengths is typically in the range $0.885 < r_0 < 0.995$. Table 3 presents a comparison between the measured *B*(*E*2) values and those we calculate using $r_0 = 0.929$fm for the 7 nuclei of interest to us here.

In the present analysis good agreement with experiment is found for $Z_2=10$ and $Z_2=12$, while for $z_2=6$ the values theory is far than experiment, since we used three different types of clusters (see figure 5-1 and figure 5-2).

The theoretical values of B(E2) (see figure 5-1) increase on average with the mass of Er isotopes. This result from the fact that the mass dependence of B(E2) is not limited to $A^{-2/3}$ .Indeed ,the reduced mass μ introduces such dependence

$$\begin{aligned} \mu &= \frac{A_2.(A-A_2)}{A} \\ &= A_2.\left(1-\frac{A_2}{A}\right) \end{aligned} \tag{II-43}$$

To reproduce the experimental values of B(E2) one must introduce a slight change in r_0 values for each Er isotope.

The choice of cluster is guided by observations of exotic decay and the value of $B(E2)$. The former strongly suggests a significant parentage of the emitted cluster in the ground state of the initial nucleus. It is particularly important to note that the large difference between the *B(E2)* results for ^{158}Er and ^{168}Er can be reproduced naturally within our model, principally through the change in cluster charge from 6 to 12 and the change in reduce masse.

The mass of the cluster is not so easily inferred, we expect the charge to mass ratio of the cluster to be similar to that of the core, to avoid setting up a large electric dipole moment.

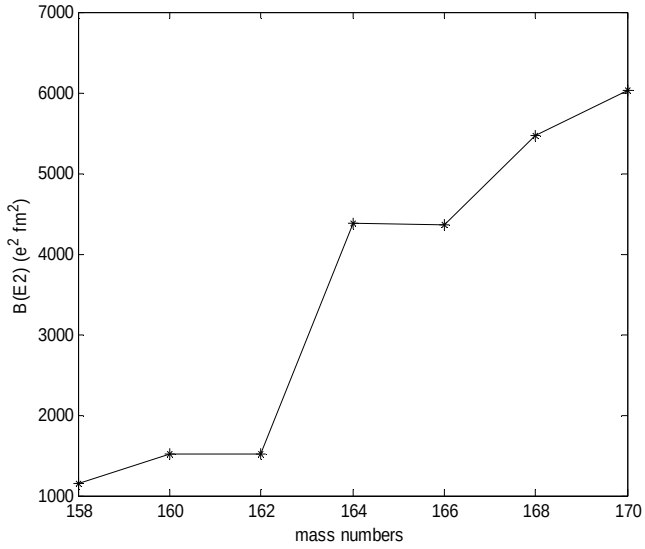


Figure 5-1. Calculations of *B(E2)* as a function of mass numbers for Er istopes.

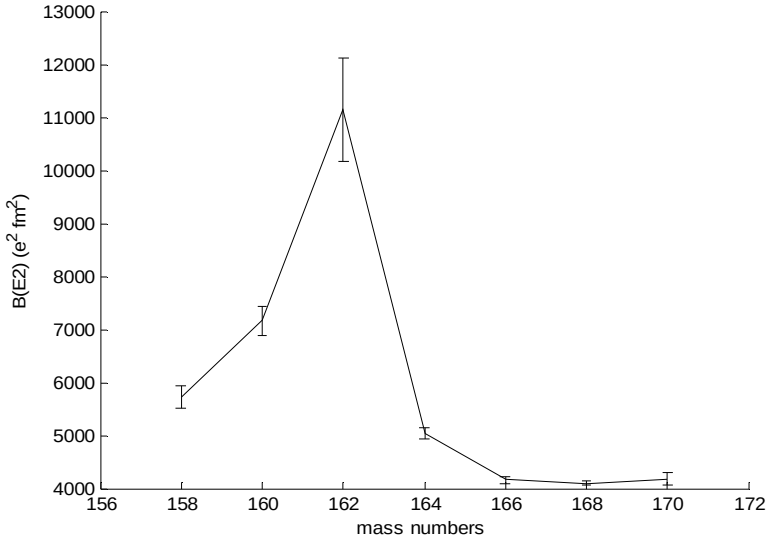


Figure 5-2. The experimental *B(E2)* as a function of mass numbers for Er istopes.

Isotop (A,Z)	$E(2^+_1)$ (KeV)	$T_{1/2}((2^+_1)$ (s)	$B_{ex}(E2)$ (e^2 fm^4)	$B_{th}(E2)$ (e^2 fm^4)
(158,68)	192.15± 0.03	$(277\pm10).10^{-12}$	5715± 210	1153
(160,68)	125.8 ±0.1	$(919\pm31). 10^{-12}$	7162±270	1518
(162,68)	102.04±0.03	$(1.17 \pm0.1).10^{-9}$	11143±968	1509
(164,68)	91.40±.02	$(1.47 \pm0.03).10^{-9}$	5046±108	4377
(166,68)	80.577±.007	$(1.82\pm0.03).10^{-9}$	4164±70	4362
(168,68)	79.804±.001	$(1.88 \pm 0.02).10^{-9}$	4097±43	5462
(170,68)	78.68±0.17	$(1.89\pm 0.03).10^{-9}$	4182±111	6024

Table 3. B(E: $2^+_1 \rightarrow 0^+_1$) in the $^{158\text{-}170}$Er even-even isotops

CHAPTER III:

Extension of the nuclear vibron model

III-1.Introduction

The vibron model has been developed to provide an algebraic method for treating diatomic molecular spectroscopy. Ideed, the spectra of such molecules are dominated by a dipole degree of freedom. The essential features of diatomic molecules that are not placed in an external electric or magnetic fields can be characterized in terms of the distance vectors $\vec{R}$ (collective variable) between the two atoms. Quantization of this degree of freedom leads to an algebraic model with four boson operators divided into a scalar s with angular momentum and parity $J^P=0^+$, and vector p bosons with quantum numbers $J^P=1^-$. The group structure of this model is that of U(4).

III-2.The U(4) algebra

The U(4) algebra can be realized in terms of four creation and for four annihilation operators b_i^+ and b_i. These components are associated with a vector operator (with components p_m^+) and scalar (rank-0) operators s^+.
The general form of bilinear operators of U(4) algebra is

$$G_i^j = b_i^+ b_j \tag{III-1}$$

where one uses the notation

$$b_n^+ = p_{n-2}^+ \quad \text{for} \quad n=1,\ldots,3 \quad \text{and} \quad b_4^+ \equiv s^+ \tag{III-2-a}$$

$$b_n = p_{n-2} \quad \text{for} \quad n=1,\ldots,3 \quad \text{and} \quad b_4 = s \tag{III-2-b}$$

The commutation relations of *s*- and *p* operators are

$$[p_\upsilon , p^+_\nu]=\delta_{\nu\upsilon} \tag{III-3-a}$$

and

$$[s, s^+]=1 \tag{III-3-a}$$

The bilinear operator Eq.(III-1) is satisfy the $U(4)$ commutation relation

$$[G_i^j, G_k^l] = G_i^l \delta_{jk} - G_k^j \delta_{il} \tag{III-4}$$

We can also define the generator of this algebra in a coupled form [18]

$$B_u^{(\lambda)}(l,l') = [b_l^+ \times \tilde{b}_{l'}]_u^{(\lambda)} = \sum_{mm'} \langle lm\, l'm' | \lambda u \rangle b_{lm}^+ \tilde{b}_{l'm'} \tag{III-5}$$

where as usual , $\tilde{p}_u = (-1)^{1-u} p_{-u}$ and $\tilde{s} = s$, or in compact form

$$\tilde{b}_{lm} = (-1)^{l-m} b_{l-m} \tag{III-6}$$

to ensures that the boson annihilation operators transform like spherical tensors.

III-3- Chain of subalgebra of U(4)

The $U(4)$ Algebra has two rotationally invariant subalgebra chains, namely [19]

1- $$U(4) \supset U(3) \supset SU(3) \supset SO(3) \supset SO(2) \tag{III-7-a}$$

2- $$U(4) \supset SO(4) \supset SO(3) \supset SO(2) \tag{III-7-b}$$

III-4.Basic operators

The linear Casimir invariant $C_1[U(n)]$ is defined as

$$C_1[U(n)] = \sum_{i\ j=1}^{n} G_i^i \tag{III-8}$$

The quadratic Casimir invariant $C_2[U(n)]$ can be written

$$C_2[U(n)] = \sum_{i=1}^{n} G_i^j G_j^i \tag{III-9}$$

The dipole and angular momentum operators are defined by [17]

$$\hat{D}_\mu = i[p^+ \times \tilde{s} - s^+ \times \tilde{p}]_\mu^{(1)} \tag{III-10-a}$$

$$\hat{L}_\mu = \sqrt{2}\left[p^+ \times \tilde{p}\right]_\mu^{(1)} \tag{III-10-b}$$

A second dipole operator is also defined as

$$\hat{D}'_\mu = [p^+ \times \tilde{s} - s^+ \times \tilde{p}]_\mu^{(1)} \tag{III-10-c}$$

The quadrupole operators is obtained as the follows

$$\hat{Q}_u = \left[p^+ \times \tilde{p}\right]^{(2)}_u \tag{III-11}$$

and

$$Q^2 = \sum_u (-1)^u Q_u Q_{-u} \tag{III-12}$$

The quadratic Casimir invariant of *SO(4)* is defined in the follows form as

$$C_2[SO(4)] = \hat{L}^2 + \hat{D}^2 \tag{III-13}$$

The quadratic Casimir operator of SO(3) is given as the follows

$$C_2[SO(3)] = \hat{L}^2 \tag{III-14}$$

The number of bosons is obtained as the expectation value of a general one-body operator

$$\hat{n}_l = \sum_m b^+_{lm} b_{lm} \tag{III-15}$$

This gives for both bosons

$$\hat{n}_p = \left[p^+ \times \tilde{p}\right]^{(1)}_u \quad , \quad \hat{n}_s = \left[s^+ \times \tilde{s}\right]^{(0)}_0 \tag{III-16}$$

The total boson-number operator is given

$$\hat{N} = \sum_k p^+_k p_k + s^+ s = \hat{n}_p + \hat{n}_s \tag{III-17}$$

The Casimir invariant of *U*(3) is

$$C_1[U(3)] = \sum_{i\ j=1}^{3} G^i_i = \sum_m p^+_m p_m = \hat{n}_p \tag{III-18}$$

The quadratic Casimir invariant of *U(3)* is given by

$$\begin{aligned} C_2[U(3)] &= \sum_{mm'} (-)^{m+m'} p_m^+ \tilde{p}_{-m'} p_{m'}^+ \tilde{p}_{-m} \\ &= \sum_{\lambda\mu} (-)^{\mu} \left[p^+ \times \tilde{p}\right]_{\mu}^{(\lambda)} \left[p^+ \times \tilde{p}\right]_{-\mu}^{(\lambda)} \\ &= \sum_{\lambda\mu} \left[p^+ \times \tilde{p}\right]_{\mu}^{(\lambda)} . \left[p^+ \times \tilde{p}\right]_{-\mu}^{(\lambda)} \\ &= \frac{1}{3} \hat{n}_p{}^2 + \frac{1}{2} \hat{L}^2 + \hat{Q}^2 \end{aligned} \qquad \text{(III-19)}$$

For symmetric representation it reduces simply to $\hat{n}_p(\hat{n}_p + 2)$.

The quadratic Casimir invariant of $U(4)$ is given by

$$\begin{aligned} C_2[U(4)] &= \sum_{ll'} (-)^{l+l'} \sum_{\lambda} B^{(\lambda)}(l,l') . B^{(\lambda)}(l',l) \\ &= \frac{1}{3} \hat{n}_p{}^2 + \frac{1}{2} (\hat{L}^2 + \hat{D}^2 + \hat{D}'^2) + \hat{Q}^2 + \hat{n}_s{}^2 \end{aligned} \qquad \text{(III-20)}$$

As in the case of U(3) it reduces simply to $\hat{N}(\hat{N} + 3)$.

III-5.The vibron model Hamiltonian

The algebraic approach to diatomic molecules consists of writing the most general Hamiltonian with one- and two-body interactions in terms of the 16 generators G_i^j :

$$\hat{H} = E_0 + \sum_{ij} e_{ij} b_i b_j + \sum_{ijkl} e_{ijkl} b_i^+ b_j^+ b_l b_k \qquad \text{(III-21)}$$

where E_0 is a constant and e_{ij} and e_{ijkl} are parameters related to the one –and two body matrix elements.

One of the advantages of the algebraic approach is that it allows one to find all possible analytic solution to the eigenvalue problem for H Eq.(III-21).

The hermiticity condition and rotational invariance of the Hamiltonian substantially reduce the number of independent parameters. We find

$$\hat{H} = E_0 + e_s \hat{n}_s + e_p \hat{n}_p + e_1 \left[\left[p^+ \times p^+ \right]^{(0)} \times \left[\tilde{p} \times \tilde{p} \right]^{(0)} \right]_0^{(0)} + e_2 \left[\left[p^+ \times p^+ \right]^{(2)} \times \left[\tilde{p} \times \tilde{p} \right]^{(2)} \right]_0^{(0)} +$$
$$e_3 \left[\left[p^+ \times p^+ \right]^{(0)} \times \left[\tilde{s} \times \tilde{s} \right]^{(0)} \right]_0^{(0)} + e_4 \left[\left[s^+ \times s^+ \right]^{(0)} \times \left[\tilde{p} \times \tilde{p} \right]^{(0)} \right]_0^{(0)} + \quad \text{(III-22)}$$
$$e_5 \left[\left[p^+ \times s^+ \right]^{(1)} \times \left[\tilde{p} \times \tilde{s} \right]^{(1)} \right]_0^{(0)} + e_6 \left[\left[s^+ \times s^+ \right]^{(0)} \times \left[\tilde{s} \times \tilde{s} \right]^{(0)} \right]_0^{(0)}$$

One expressing the vibron Hamiltonian Eq(III-22) in terms of the Casmir invariant operators in the form

$$\hat{H} = E_0' + e_1'' \hat{n}_p + e_2'' \hat{n}_p^2 + e_3'' C_2[SO(4)] + e_4'' C_2[SO(3)] \quad \text{(III-23)}$$

One can express the vibron Hamiltonian Eq.(III-23) in terms of the invariant operators

$$\hat{H} = E_0'' + e_1' \hat{n}_p + e_2' \hat{n}_p^2 + e_3' \hat{L}^2 + e_4' \hat{D}^2$$

(III-24)
The compact U(4) algebra also indicates that the spectra are discrete and that the numbers of the states is finite while the non compact of this algebra indicates that the spectra are continuous .

III-6 Dynamical symmetries

The Hamiltonian describing the boson system is written in terms of the Casimir operators in one of the chains, the eigenvalue problem can be solved analytically.

III-6-1.The U(3) limit

The Hamiltonians in the $U(3)$ symmetry limits of the vibron model can be written in the multipole form by taking e_4'=0 .

$$\hat{H}_1 = E_0' + e_1' \hat{n}_p + e_2' \hat{n}_p^2 + e_3' \hat{L}^2 \quad \text{(III-25)}$$

The explicit form of $U(3)$ sates:

$$\left| [N] n_p L M_L \right\rangle \quad \text{(III-26)}$$

where N, n_p, L, and M_L label the irreducible representation of $U(4)$, $U(3)$, $SO(3)$, and $SO(2)$, respectively.

The states are eigenstates of the following operators:

$$\begin{aligned}
&\hat{N}\left|[N]n_p LM_L\right\rangle = N\left|[N]n_p LM_L\right\rangle \\
&\hat{n}_p\left|[N]n_p LM_L\right\rangle = n_p\left|[N]n_p LM_L\right\rangle \\
&\hat{L}^2\left|[N]n_p LM_L\right\rangle = L(L+1)\left|[N]n_p LM_L\right\rangle \\
&\hat{L}_0\left|[N]n_p LM_L\right\rangle = M_L\left|[N]n_p LM_L\right\rangle \\
&\hat{n}_s\left|[N]n_p LM_L\right\rangle = n_s\left|[N]n_p LM_L\right\rangle
\end{aligned} \tag{III-27}$$

The $U(4) \supset U(3)$ reduction rule takes the form

$$n_p = 0,1, \ldots, N \tag{III-28}$$

while the reduction rule associated whith $U(3) \supset SO(3)$ is given by

$$L = n_p, n_p - 2, \ldots, 1 \text{ or } 0 \tag{III-29}$$

The Hamiltonian H_1 has the eigenvalues [6]

$$E_1 = E_0 + e_1 n_p + e_2 n_p^2 + e_3 L(L+1) \tag{III-30}$$

A schematic example of eigenspectrum of $\hat{H}_1$ is shown in figure (III-1)

The reduction from totally symmetric of the irreducible representation of $U(3)$ is

$$[N] \supset (N,0) \oplus (N-1,0) \oplus \ldots\ldots \oplus (1,0) \oplus (0,0) \tag{III-31}$$

This readuction leads only to the totally symmetric of the irreducible representation of $SU(3)$

The $U(3)$ Limit leads to a vibrational spectrum (harmonic oscillator) and is not of much relevance to molecules.

III-6-2.The SO(4) limit

The Hamiltonians in the $SO(4)$ symmetry limits (III-7-b) of the vibron model can be defined as

$$\hat{H}_2 = E_0'' + e_3' \hat{L}^2 + e_4' \hat{D}^2 \tag{III-32}$$

In this case one can write the Hamiltonian (III-32) in the terms of Casimir operators

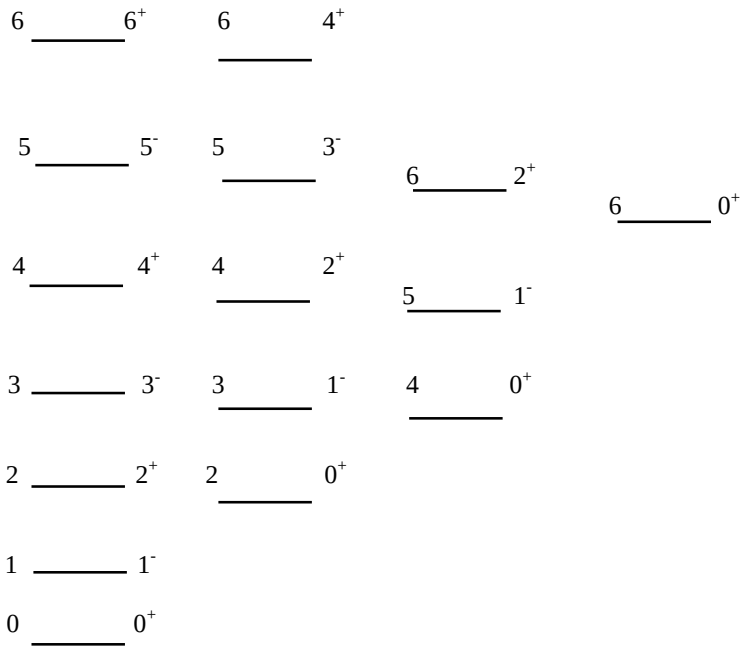


Figure III-1.The typical eigenspectrum for U(3) limit for *N*=6

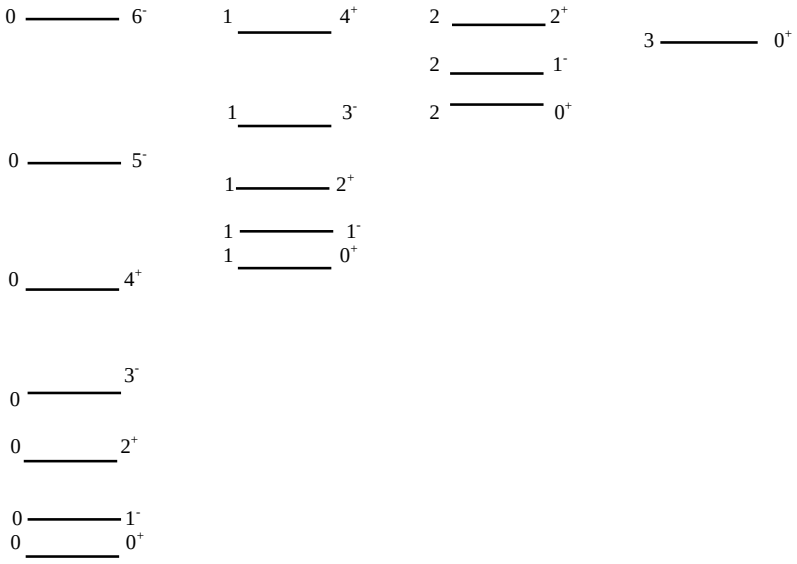


Figure III-2.The typycal eigenspectrum for SO(4) limit for *N*=6

$$\hat{H} = E_0' + e_3'' C_2[SO(4)] + e_4'' C_2[SO(3)] \tag{III-33}$$

The *SO*(4) limit, on the other hand, leads to a spectrum similar to that of a Morse potential and hence it is appropriate for the description of the rotation-vibration spectra of molecules.
The states in the *SO*(4) limit are thus characterized by the quantum numbers N, τ ,L and M_L and are denoted $\left|[N]\tau LM_L\right\rangle$.

where N, τ, L , and M_L label the irreducible representation of $U(4)$, $SO(4)$, $SO(3)$, and $SO(2)$, respectively.
The operator equations defining the *SO*(4)

$$\begin{aligned}
&\hat{N}\left|[N]\tau LM_L\right\rangle = N\left|[N]\tau LM_L\right\rangle \\
&(\hat{L}^2 + \hat{D}^2)\left|[N]\tau LM_L\right\rangle = \tau(\tau+2)\left|[N]\tau LM_L\right\rangle \\
&\hat{L}^2\left|[N]\tau LM_L\right\rangle = L(L+1)\left|[N]\tau LM_L\right\rangle \\
&\hat{L}_0\left|[N]\tau LM_L\right\rangle = M_L\left|[N]\tau LM_L\right\rangle
\end{aligned} \tag{III-34}$$

The number quantum τ is detrmined by $U(4) \supset SO(4)$ branching rules

$$\tau = N, N-2, \ldots, 1 \text{ or } 0 \tag{III-35}$$

The allowed values of L are obtained from $SO(4) \supset SO(3)$ branching rules

$$L = 0, 1, \ldots, \tau \tag{III-36}$$

The eigenenergy of the Hamiltonian Eq.(III-33)can be represented this way

$$E_2 = E_0 + e'_1\,\tau(\tau+2) + e'_2\,L(L+1) \tag{III-37}$$

The eigenvalue expression (III-37) completely determines the spectrum of Hamiltonian $\hat{H}_2$ is shown in figure (III-2).
In its simplest form with a one-body dipole operator, the *SO*(4) limit corresponds to a rigid rotor with vanishing vibrational transitions.

III-7.Algebraic nuclear cluster model $U(6)\otimes U(4)$

the irreducible representation of $U(4)$ can be classified by labels $[M_a, M_b, M_c, M_d]$ while the totally symmetric irreducible representation of $U(4)$ is labelled by $[M, 0, 0, 0]$. α-particles can treated the by incorporate them together into large cluster, this assumption leads to reduces the group structure to follow group

$$G = U(6)\otimes U(4) \tag{III-38}$$

The representation of the group G are considered with conserve the total number of the bosons

$$T=N+M \tag{III-39}$$

N is the number of bosons in representation of $U(6)$ and M label the totally symmetric irreducible representation of $U(4)$ which is be chosen equal the number of particle bosons removed to form α-clusters. For no α-clusters $M=0$, for one α-clusters $M=2$, for tow α-clusters $M=4$, etc. The number N and M are each conserved in any configuration of α-clusters [21].

III-7-1.The $U(6)\otimes U(4)$ algebra

For simplicity, in this note one neglect the differences between proton and neutron bosons since this does not involve the simple phenomenological considerations which follow.

Discussing an algebraic treatment of tow-cluster systems, in which the eigenvalue problem is solved by matrix diagonalization instead of by solving a set of differential equations. In the algebraic cluster model (ACM), the method of bosonic quantization is used. The $U(6)\otimes U(4)$ Lie algebra, spanned by operators

$$a^{+}_{lm} = \left\{s^{+}, d^{+}_{m}\right\} \quad \text{for } l=0,2 \tag{III-40}$$

which generate the algebra of $U(6)$ and four boson operators

$$b^{+}_{lm} = \left\{s'^{+}, p^{+}_{m}\right\} \quad \text{for } l = 0,1 \tag{III-41}$$

which generate the algebra of $U(4)$. With commutation relations

$$\begin{aligned}
&\left[a_{lm}, a^{+}_{l'm'}\right] = \delta_{ll'}\delta_{mm'} \\
&\left[a_{lm}, a_{l'm'}\right] = \left[a^{+}_{lm}, a^{+}_{l'm'}\right] = 0 \\
&\left[b_{lm}, b^{+}_{l'm'}\right] = \delta_{ll'}\delta_{mm'} \\
&\left[b_{lm}, b_{l'm'}\right] = \left[b^{+}_{lm}, b^{+}_{l'm'}\right] = 0
\end{aligned} \tag{III-42}$$

Then all *a*-bosons commute with the *b*-bosons. The set of 36 bilinear products of creation and annihilation operators spans the Lie algebra of $U(6)$

$$G^{a}_{lm,l'm'} = a^{+}_{lm}\tilde{a}_{l'm'} \qquad \text{for } l,l'=0,2 \tag{III-43}$$

while the 16 bilinear products

$$G^{b}_{lm,l'm'} = b^{+}_{lm}\tilde{b}_{l'm'} \qquad \text{for } l,l'=0,1 \tag{III-44}$$

Spans the Lie algebra of $U(4)$ in that case $U(6)\otimes U(4)$ algebra can be generated from the 52 bilinear products. The form of this bilinear operator is

$$G = \left\{G^{a}, G^{b}\right\} \tag{III-45}$$

The bilinear operator is satisfy the $U(6)\otimes U(4)$ commutation relation

$$[G^{j}_{i}, G^{l}_{k}] = G^{l}_{i}\delta_{jk} - G^{j}_{k}\delta_{il} \tag{III-46}$$

can also define the generator of the lie algebra associated with this algebra .This generators will be written as the $O(3)$ tensors

$$\begin{aligned}
A^{(\lambda)}_{u}(l,l') &= [a^{+}_{l} \times \tilde{a}_{l'}]^{(\lambda)}_{u} \\
&= \sum_{mm'} \left\langle lm\, l'm' \middle| \lambda u \right\rangle a^{+}_{lm}\tilde{a}_{l'm'}
\end{aligned} \tag{III-47}$$

Generate the Lie algebra of $U(6)$,and

$$\begin{aligned}
B^{(\lambda)}_{u}(l,l') &= [b^{+}_{l} \times \tilde{b}_{l'}]^{(\lambda)}_{u} \\
&= \sum_{mm'} \left\langle lm\, l'm' \middle| \lambda u \right\rangle b^{+}_{lm} b_{l'm'}
\end{aligned} \tag{III-48}$$

generate the Lie algebra of $U(4)$.

The commutation relation is

$$\left[A_k^{(K)}(l,l'),A_{k'}^{(K')}(l,l')\right]=\sum_{K'',k''}(2k+1)^{1/2}(2k'+1)^{1/2}\left\langle kKk'K'\middle|k''K''\right\rangle$$
$$\times(-1)^{k-k'}\left[(-1)^{k+k+k''}\begin{Bmatrix}kk'k''\\l'''ll'\end{Bmatrix}\delta_{l'l''}A_u^{(\lambda)}(l,l')-\begin{Bmatrix}kk'k''\\l''l'l\end{Bmatrix}\delta_{l'l''}A_u^{(\lambda)}(l,l')\right] \quad \text{(III-49)}$$

And

$$\left[A_k^{(K)}(l,l'),B_{k'}^{(K')}(l'',l''')\right]=0 \quad \text{(III-50)}$$

From this relation can determine the structure constants of the $U(6)\otimes U(4)$.algebra .

III-7-2.The $U(6)\otimes U(4)$ Hamiltonian

The Hamiltonian in the $U(6)\otimes U(4)$ algebra. It can be written as[22]

$$H=H_a+H_b+H_{ab} \quad \text{(III-51)}$$

H_a descript the Hamiltonian in the *U(*6*)* algebra in the *s-d* boson spanned for one and two-body interactions (see Eq.(I-98)), while H_b is the Hamiltonian in the *U(4)* algebra in the *s-p* boson spanned for one and two-body interactions (see Eq. (III-22)).

In general, this Hamiltonian must be diagonalized numerically .there are however simple cases for which the problem can be solved analytically.

H_{ab} are the interactions between *a*-bosons and *b*-bosons. which is obtained by the fallow relation. In the H_{ab} is no one –body interaction. All interactions H_{ab} which are n-body with n-odd vanish.

$$
\begin{aligned}
\hat{H}_{ab} &= +e_1\Big[\Big[s^+\times s'^+\Big]^{(0)}\times\big[\tilde{s}\times\tilde{s}'\big]^{(0)}\Big]_0^{(0)}+\Big[\Big[s'^+\times s^+\Big]^{(0)}\times\big[\tilde{s}'\times\tilde{s}\big]^{(0)}\Big]_0^{(0)}\\
&+e_4\Big[\Big[d^+\times p^+\Big]^{(1)}\times\big[\tilde{d}\times\tilde{p}\big]^{(1)}+\Big[p^+\times d^+\Big]^{(1)}\times\big[\tilde{p}\times\tilde{d}\big]^{(1)}\Big]_0^{(0)}\\
&+e_5\Big[\Big[d^+\times p^+\Big]^{(2)}\times\big[\tilde{d}\times\tilde{p}\big]^{(2)}+\Big[p^+\times d^+\Big]^{(2)}\times\big[\tilde{p}\times\tilde{d}\big]^{(2)}\Big]_0^{(0)}\\
&+e_6\Big[\Big[d^+\times p^+\Big]^{(3)}\times\big[\tilde{d}\times\tilde{p}\big]^{(3)}+\Big[p^+\times d^+\Big]^{(3)}\times\big[\tilde{p}\times\tilde{d}\big]^{(3)}\Big]_0^{(0)}\\
&+e_7\Big[\Big[s^+\times p^+\Big]^{(1)}\times\big[\tilde{d}\times\tilde{p}\big]^{(1)}+\Big[p^+\times d^+\Big]^{(1)}\times\big[\tilde{p}\times\tilde{s}\big]^{(1)}\Big]_0^{(0)}\\
&+e_8\Big[\Big[s^+\times p^+\Big]^{(1)}\times\big[\tilde{s}\times\tilde{p}\big]^{(1)}+\Big[p^+\times s^+\Big]^{(1)}\times\big[\tilde{p}\times\tilde{s}\big]^{(1)}\Big]_0^{(0)}\\
&+e_9\Big[\Big[d^+\times s'^+\Big]^{(2)}\times\big[\tilde{d}\times\tilde{s}'\big]^{(2)}+\Big[s'^+\times d^+\Big]^{(2)}\times\big[\tilde{s}\times\tilde{d}\big]^{(2)}\Big]_0^{(0)}
\end{aligned}
\qquad \text{(III-52)}
$$

There are, in general, nine H_a terms, seven H_b terms, and seven H_{ab} terms. Many of these are not important and can be eliminated in specific studies resulting from dynamic symmetries.

Can rewrite the most general Hamiltonian (III-51) in the terms of the generators $U(6)\otimes U(4)$ as fallow .

$$
\begin{aligned}
H &= E_a + E_b + \sum_{lm,l'm'} a_{lm,l'm'}G^a_{lm,l'm'} + \sum_{lm,l'm'} b_{kn,k'n'}G^b_{kn,k'n}\\
&+\frac{1}{2}\sum_{\substack{m,l'm,\\ l''m'',l'''m''}} A_{m,l'm,l''m'',l'''m'''}G^a_{lm,l'm'}G^a_{l''m'',l'''m'''}\\
&+\frac{1}{2}\sum_{\substack{lkn,k'n,\\ k''n'',k'''n''}} B_{lkn,k'n,k''n'',k'''n'''}G^b_{lkn,k'n'}G^b_{k''n'',k'''n'''}\\
&+\frac{1}{2}\sum_{\substack{m,l'm,\\ kn,k'n}} C_{m,l'm,kn,k'n}G^a_{lm,l'm'}G^b_{kn,k'n}
\end{aligned}
\qquad \text{(III-53)}
$$

the Hamiltonian will be as an $O(3)$ invariant .

III-7-3.Dynamical symmetries

The Hamiltonian Eq.(III-51), can be diagonalized analytically, in special case, this case is obtained by introduce the concept of dynamic symmetries of the coupled system of (*s*,*d*) and (*s'*,*p*) bosons.

The chains (I-95) and (III-7-a) have a common subgroup, *SU(3)*. For that situation one can coupled the chain (III-60-b) with chain (III-61-a) which is obtains the $SU(3)$ limit .

the dynamic symmetries of the coupled system, $U(6)\otimes\ U(4)$. Described by the group chains

$$U_a(6)\otimes U_b(4)\supset SU_a(3)\otimes U_b(3)\supset O(3)\supset O(2) \tag{III-54}$$

and

$$U_a(6)\otimes U_b(4)\supset U_a(5)\otimes U_b(3)\supset O_a(5)\otimes O_b(2)\supset O_a(3)\otimes O_b(3)\supset O(3)\supset O(2) \tag{III-55}$$

This has numerical solutions. And

$$U_a(6)\otimes U_b(4)\supset SU_a(3)\otimes U_b(3)\supset SU_a(3)\otimes SU_b(3)\supset SU(3)\supset O(3)\supset O(2) \tag{III-56}$$

This is a study by analytic diagonalization of Hamiltonian matrix in limit *SU(3)*.

III-7-3-1. $U_a(5)\otimes U_b(3)$ **Limit**

The Hamiltonian characterising the $U_a(5)\otimes U_b(3)$ limit (III-55), corresponds to

$$\begin{aligned}\hat{H}_1 = e_d\,\hat{n}_d+\alpha_d\,\hat{n}_d(\hat{n}_d-1)+e_p\,\hat{n}_p+\alpha_p\,\hat{n}_p(\hat{n}_p-1)+\\ e'_{11}\,C_2[SO(5)]+e'_{12}\,C_2[SO(3)]\end{aligned} \tag{III-57}$$

The eigenstate corresponding to this Hamiltonian are labelled by

$$\left|[N],[M],n_d,v,n_\Delta,L_d;n_p,L_p;L,M_L\right\rangle \tag{III-58}$$

And the energy eigenvalues are given by

$$E_1 = e_d n_d+\alpha_d n_d(n_d-1)+e_p n_p+\alpha_p n_p(n_p-1)+e'_{11}\,v(v+3)+e'_{12}\,L(L+1) \tag{III-59}$$

Each vibrational state n_p=0,1,2… is split in deformed nuclei into bands, these bands correspond to different orientations of the vibrations with respect to the symmetry axis.
the case n_p =1, this state splits into tow parts with K^p=0,-,1-.the first corresponds to vibrations along the symmetry axis, while the seconds corresponds to vibrations perpendicular to symmetry axis.

III-7-3-2.SU(3) limit

The Hamiltonian characterizing the *SU(3)* limit (III-66), corresponds to

$$\hat{H}_1 = e'_{12}\,C_1[U_b(3)]+e'_{11}\,C_2[SU_b(3)]+e'C_2[SU_a(3)]+e'_9\,C_2[SU(3)]+e'_{10}\,C_2[SO(3)] \tag{III-60}$$

Writing 9 generators of $U_b(3)$algebra as

$$B_0^{(1)}(1,1) = \left[p^+ \times \tilde{p}\right]_0^{(1)} \tag{III-61-a}$$

$$B_u^{(1)}(1,1) = \sqrt{2}\left[p^+ \times \tilde{p}\right]_\mu^{(1)} = L_u^b \tag{III-61-b}$$

$$B_u^{(2)}(1,1) = \frac{\sqrt{3}}{2}\left[p^+ \times \tilde{p}\right]_\mu^{(2)} = Q_u^b \tag{III-61-c}$$

The linear Casimir invariant of $U_b(3)$ is given by the *d* boson number operator as

$$C_1[U_b(3)] = \sum_m p_m^+ p_m = \hat{n}_p \tag{III-62}$$

The eight generators of $SU_a(3)$ are

$$\hat{L}_u = A_u^{(1)}(2,2) = \sqrt{10}[d^+ \times \tilde{d}]_u^{(1)} \tag{III-63}$$

and

$$\begin{aligned} A_u^{(2)}(0,2) &= Q_u \\ &= \left[d^+ \times s\right]^{(2)}{}_u + \left[s^+ \times \tilde{d}\right]^{(2)}{}_u - (\sqrt{7}/2)\left[d^+ \times \tilde{d}\right]^{(2)}{}_u \end{aligned} \tag{III-64}$$

And quadratic Casimir invariant of $SU_a(3)$ is obtained

$$C_2[SU_a(3)] = 2\hat{Q}^2 + \frac{3}{4}\hat{L}^2 \tag{III-65}$$

The eight generators of the combined *SU*(3) group are

$$C_u^{(1)} = L_u = \sqrt{10}[d^+ \times \tilde{d}]_u^{(1)} \tag{III-66}$$

$$C_u^{(2)} = Q_u = \left[d^+ \times s\right]^{(2)}{}_u + \left[s^+ \times \tilde{d}\right]^{(2)}{}_u \pm (\sqrt{7}/2)\left[d^+ \times \tilde{d}\right]^{(2)}{}_u \pm \left(\sqrt{3}/2\right)\left[p^+ \times \tilde{p}\right]_k^{(2)} \tag{III-67}$$

The generators of *SO*(3) are

$$\sqrt{10}[d^+ \times \tilde{d}]_u^{(1)} \tag{III-68}$$

The quadratic Casimir invariant of *SO*(3) is obtained

$$C_2[SO(3)] = \hat{L}^2 \tag{III-69}$$

The states of $U(6) \otimes U(4)$ algebra may be represented in the form [19] :

$$\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \tag{III-70}$$

where N, M,$(\lambda,\mu)_a$ (λ,μ) , L, and M_L label the irreducible representation of $U(6)$, $U(4)$, $SU_a(3)$, $SU(3)$,$SO(3)$ and $SO(2)$, respectively. $(n_p,0)_b$ label the totally symmetric of the $SU_b(3)$. The k is the $SU(3) \supset SO(3)$ label introduced by Verdagos .

The states $\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle$ are eigenstates of the following operators:

$$
\begin{aligned}
&C_1[U_b(3)]\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&\qquad\qquad n_p \;=\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&C_2[SU_a(3)]\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&\qquad\qquad C(\lambda_a,\mu_a)=\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&C_2[SU_b(3)]\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda_b,\mu_b)k;L,M_L\right\rangle \\
&\qquad\qquad C(\lambda_b,\mu_b)=\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&C_2[SU(3)]\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&\qquad\qquad C(\lambda,\mu)=\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&C_2[SO(3)]\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle \\
&\qquad\qquad L(L+1)=\left|[N],[M],(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)k;L,M_L\right\rangle
\end{aligned}
\qquad \text{(III-71)}
$$

where

$$C(\lambda,\mu)=\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu) \qquad \text{(III-72)}$$

is the eigenvalue of the quadratic Casimir invariant of SU(3).

The $U_b(4) \supset U_b(3)$ reduction rule takes the form

$$n_p=0,1,\ldots,M \qquad \text{(III-73)}$$

The rule which is needed for constructing the coupled states of the limits $SU(3)$is

$$(\lambda,\mu)\otimes(1,0)=(\lambda+1,\mu)\oplus(\lambda-1,\mu+1)\oplus(\lambda,\mu-1) \qquad \text{(III-74)}$$

When λ or μ equal to zero.

The eigenenergy of $SU(3)$ limit can be represented in this way by the follow formula [22]

$$E=e_1n_p+e_2n_p(n_p+3)+e_3C(\lambda_a,\mu_a)+e_4C(\lambda,\mu)+e_5L(L+1) \qquad \text{(III-75)}$$

The *SU*(3) limit describes oscillations of a spherical cluster (for example an α-particle) in a nucleus with axially symmetric deformation. It is interesting to compare this situation with that of oscillation of the cluster in a spherical nucleus.

The structure of the spectrum of α-cluster (see Eq. (III-75) and fig(III-3)) are depends on the angular momenta of the elementary modes of excitation from which the α-cluster are built. One expects modes with $\boldsymbol{J}^P=0^+,1^-,2^+,3^-,\ldots$.in the normal configurations, the $\boldsymbol{J}^P=1^-$ mode is absent since it corresponds to a displacements of the centre of mass. thus ,one usually truncates the angular momentum of the normal modes to

$$\boldsymbol{J}^P=0^+,2^+,4^+,\ldots. \qquad \text{(III-76)}$$

If the nucleus has an axially symmetric quadrupole deformation, the coupling of additional degrees of freedom occurs through a quadrupole-quadrupole interaction. This interaction is of the form

$$V_{ab} = k_{ab} Q_a . Q_b \qquad \text{(III-77)}$$

where

$$C_2[SU(3)] = 2Q.Q + 3/4 L.L \qquad \text{(III-78)}$$

and

$$Q = Q_a + Q_b \qquad \text{(III-79-a)}$$

$$L = L_a + L_b \qquad \text{(III-79-b)}$$

Replacing this expression in Eq (III-77); one finds that

$$V_{ab} = k_{ab} Q_a . Q_b \left(C_2[SU(3)] - C_2[SU_a(3)] - C_2[SU_b(3)] \right) - \frac{3}{2} L_a . L_b \qquad \text{(III-80)}$$

Another form of the interaction V_{ab}, is writing as fallows:

$$V_{ab} = k'_{ab} L_a . L_b \qquad \text{(III-81)}$$

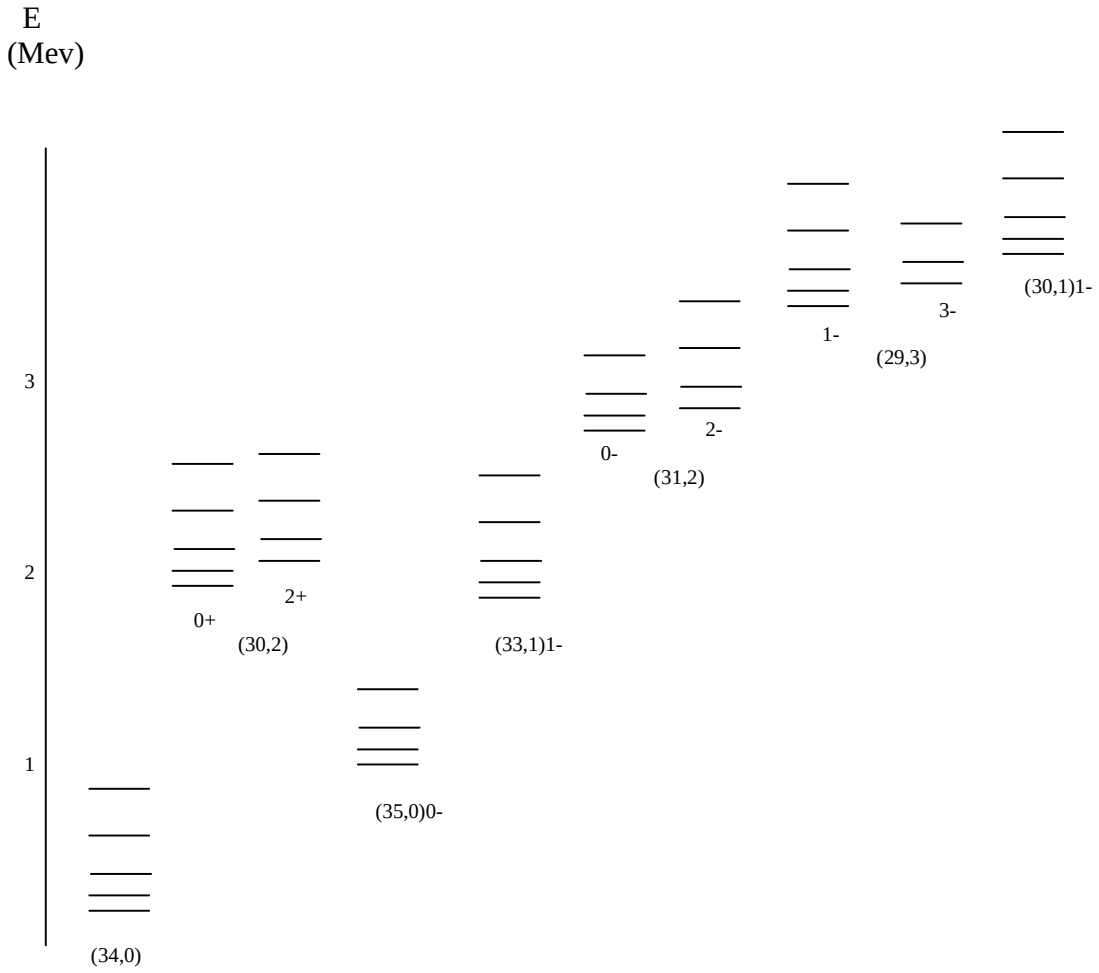


Figure III-3.The typycal eigenspectrum for SU(3) limit for N_T=19 in (1α) configuration .Bands are labelled by the quantum number (λ,μ)χP

III-7-4.Transition Operators

The most general one-body electromagnetic transition operator in terms of creation and annihilation operators has the form

$$T_1^{(L)} = \sum_{\alpha\beta} c_{\alpha\beta}^{(L)} b_\alpha^+ b_\beta \qquad \text{(III-82)}$$

These, when written in coupled tensor form, read [6]

$$T_{1,u}^{(L)} = \sum_{\alpha\beta,l,l'} c_{\alpha\beta}^{(L)} [b_l^+ \times \tilde{b}_{l'}]_u^{(L)} \tag{III-83}$$

Assuming that

$$b_{lm}^+ = \left\{ s^+, p_m^+, d_m^+ \right\} \quad (l = 0, 1, 2) \tag{III-84}$$

In $U(6) \otimes U(4)$ model , writing the operator in term of *s* and *p* and *d* as

$$T(E_0) = a[s^+ \times \tilde{s}]_0^{(0)} + b[d^+ \times \tilde{d}]_0^{(0)} + c[s'^+ \times \tilde{s}']_0^{(0)} + d[p^+ \times \tilde{p}]_0^{(0)} \tag{III-85-a}$$

$$T(E_1) = a_1[p^+ \times \tilde{s}' + s'^+ \times \tilde{p}]_m^{(1)} \tag{III-85-b}$$

$$T(M_1) = a_2[d^+ \times \tilde{d}]_0^{(0)} + b_2[p^+ \times \tilde{p}]_0^{(0)} \tag{III-85-c}$$

$$T(E_2) = a_3[d^+ \times \tilde{s} + s^+ \times \tilde{d}]_m^{(2)} + c_3[d^+ \times \tilde{d}]_m^{(2)} + d_3[p^+ \times \tilde{p}]_m^{(2)} \tag{III-85-d}$$

$$T(M_3) = a_3[d^+ \times \tilde{d}]_m^{(3)} \tag{III-85-e}$$

$$T(E4) = a_4[d^+ \times \tilde{d}]_m^{(4)} \tag{III-85-f}$$

In writing Eq(III-85) the *s* and *d* bosons have positive parity, and *p* bosons have negative parity. Thus, for example, the operator with multipolarity one has positive parity and therefore corresponds to an *M*1 operator and the *E*1 operator would have negative parity.

III-8 Effect of the higher order terms

III-8-1 Higher order terms in the Hamiltonian

In our work, we consider an SU(3) dynamical symmetry Hamiltonian constructed from the second ,third and fourth order invariant operators of the $SU(3) \supset SO(3)$ basis [24]

$$H = H_0 + aC_2 + bC_3 + c\Omega + d\Lambda \tag{III-86}$$

The basic operators used are

$$L_\mu^b = \sqrt{2}\left[p^+ \times \tilde{p}\right]_\mu^{(1)} \tag{III-87-a}$$

$$Q_\mu^b = \frac{\sqrt{3}}{2}\left[p^+ \times \tilde{p}\right]_\mu^{(2)} \tag{III-87-b}$$

and

$$C_2[SU(3)] = 2\hat{Q}^2 + \frac{3}{4}\hat{L}^2 \tag{III-88-a}$$

$$C_3 = C_3[SU(3)]$$
$$= -\frac{4}{9}\sqrt{35}\left[Q\times Q\times Q\right]_0^{(0)} - \frac{\sqrt{15}}{2}\left[L\times Q\times L\right]_0^{(0)} \quad \text{(III-88-b)}$$

$$\Lambda = \left[L\times Q\times Q\times L\right]_0^{(0)} \quad \text{(III-88-c)}$$

$$\Omega = -3\sqrt{\frac{5}{2}}\left[L\times Q\times L\right]_0^{(0)} \quad \text{(III-88-d)}$$

The operators Λ and Ω are not diagonal in the Elliott basis. Furthermore, they do not commute. Eigenvalues have been calculated by several methods, both numerically and analytically.

In this work will be considered high order terms in *s* and *p* only. The Hamiltonians in this case can be written in following form

$$H = H_{SU(3)} + aC_3[SU(3)] + b\Lambda + c\Omega \quad \text{(III-89)}$$

We defined *H'* as following

$$H' = aC_3[SU(3)] + b\Lambda + c\Omega \quad \text{(III-90)}$$

We can considered *H'* as a perturbation term .Using perturbation theory we can obtain

$$E^{(1)} = E + \langle\varphi|H'|\varphi\rangle \quad \text{(III-91)}$$

$$|\varphi\rangle \equiv |(\lambda,\mu)_a;(n_p,0)_b;(\lambda,\mu)\chi;L,M_L\rangle \quad \text{(III-92)}$$

$$|\varphi_i\rangle \equiv |(\lambda_i,\mu_i)_a;(n_{pi},0)_b;(\lambda_i,\mu_i)\chi_i;L_i,M_{Li}\rangle \quad \text{(III-93)}$$

Here the following notation is used

$$Q_u^{(2)} = \left[p^+\times p\right]_u^{(2)} \quad \text{(III-94-a)}$$
$$\Omega = -3\sqrt{\frac{5}{2}}\left[L\times Q\times L\right]_0^{(0)} \quad \text{(III-94-b)}$$
$$\Lambda = \left[L\times Q\times Q\times L\right]_0^{(0)} \quad \text{(III-94-c)}$$

One rewrites the wave functions Eq(III-70) as

$$\left|(\lambda,\mu)_a,\chi_a;(n_p,0)_b,(\lambda,\mu);\chi L,M_L\right\rangle=\sum_{\substack{\chi_a,l_a,m_a\\ l_b,m_b}}\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b\left|(\lambda,\mu);\chi,L,M_L\right.\right\rangle \left|(\lambda,\mu)_a,\chi_a,l_a,m_a\right\rangle\times\left|(n_p,0)_b,l_b,m_b\right\rangle \tag{III-95}$$

The Racah factorization lemma can then be used to factor this coupling coefficient into two terms [25]

$$\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b\left|(\lambda,\mu),\chi,L,M_L\right.\right\rangle= \left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b\right\|(\lambda,\mu),\chi,L\rangle\langle l_a,m_a;l_b,m_b\left|L,M_L\right.\rangle \tag{III-96}$$

i)The effect of Ω operator

Firstly we calculated the effect of Ω operator. Using the definition of the coupled operators

$$\begin{aligned}\Omega&=-3\sqrt{\frac{5}{2}}\left[L\times Q\times L\right]_0^{(0)}\\&=-3\sqrt{\frac{5}{2}}\sum\nolimits_{mu}\langle 1u1-u\,|\,00\rangle\langle 1m2u-m\,|\,1u\rangle L_m Q_{u-m}L_{-u}\end{aligned} \tag{III-97}$$

we calculated the expectation value of the operators Ω in a state $|\varphi\rangle$ as:

$$\langle\varphi|\Omega|\varphi\rangle=-3\sqrt{\frac{5}{2}}\sum\nolimits_{mu}\langle 1u1-u\,|\,00\rangle\langle 1m2u-m\,|\,1u\rangle\langle\varphi|L_m Q_{u-m}L_{-u}|\varphi\rangle \tag{III-98}$$

By using the relations:

$$\langle\varphi|L_m Q_{u-m}L_{-u}|\varphi\rangle=\sum\nolimits_{\varphi_1\varphi_2}\langle\varphi|L_m^{(1)}|\varphi_1\rangle\langle\varphi_1|Q_{u-m}^{(2)}|\varphi_2\rangle\langle\varphi_2|L_{-u}^{(1)}|\varphi\rangle \tag{III-99}$$

and

$$\begin{aligned}\left\langle\varphi\left|L_m^{(1)}\right|\varphi_1\right\rangle=\sum_{\substack{\chi_a,l_a,m_a\\ l_b,m_b\\ \chi_{a1},l_{a1},m_{a1}\\ l_{b1},m_{b1}}}&\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b\left|(\lambda,\mu);\chi,L,M_L\right.\right\rangle\\&\left\langle(\lambda,\mu)_{a1},\chi_{a1},l_{a1},m_{a1};(n_{p1},0)_{b1},l_{b1},m_{b1}\left|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\right.\right\rangle\\&\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a\left|(\lambda,\mu)_{a1},\chi_{a1},l_{a1},m_{a1}\right.\right\rangle\left\langle(n_p,0)_b,l_b,m_b\left|L_m^{(1)}\right|(n_{p1},0)_{b1},l_{b1},m_{b1}\right\rangle\end{aligned} \tag{III-100}$$

where $\left|(\lambda,\mu),\chi,l,m\right\rangle$ represents the Elliott basis.

The Elliot basis can be expanded onto the Vergados basis $\left|(\lambda,\mu),K,l,m\right\rangle$ [26] as follows

$$\left|(\lambda,\mu),\chi,l,m\right\rangle=\sum_{K}x_{\chi K}\left|(\lambda,\mu),K,l,m\right\rangle \tag{III-101}$$

One defines the overlap integrals

$$\left\langle(\lambda,\mu),\chi,l,m\right|\left.(\lambda,\mu),\chi',l,m\right\rangle=I_{KK'L} \tag{III-102}$$

The scalar product of these two bases is given by [27]

$$\begin{aligned}\left\langle(\lambda,\mu),\chi,l,m\right|\left.(\lambda,\mu),\chi',l,m\right\rangle&=\sum_{KK'}x_{\chi K}x_{\chi'K'}\left\langle(\lambda,\mu),K,l,m\right|\left.(\lambda,\mu),K',l,m\right\rangle\\&=\sum_{KK'}x_{\chi K}x_{\chi'K'}\delta_{KK'}\\&=\sum_{K}x_{\chi K}x_{\chi'K}\end{aligned} \tag{III-103}$$

The Elliott basis is not orthogonal and one thus needs the overlap integrals

$$\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a\right|\left.(\lambda,\mu)_{a1},\chi_{a1},l_{a1},m_{a1}\right\rangle=\sum_{K_a}x_{\chi K_a}x_{\chi'K_a} \tag{III-104}$$

We substitute Eq(III-104) in Eq.(III-100); we find :

$$\begin{aligned}\left\langle\varphi\right|\left.L_m^{(1)}\right|\left.\varphi_1\right\rangle=\sum_{\substack{\chi_a,K_a,l_a,m_a\\l_b,m_b\\\chi_{a1},K_{a1},l_{a1},m_{a1}\\l_{b1},m_{b1}}}&\left\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b\right|\left.(\lambda,\mu);\chi,L,M_L\right\rangle\\&\left\langle(\lambda,\mu)_{a1},\chi_{a1},l_{a1},m_{a1};(n_{p1},0)_{b1},l_{b1},m_{b1}\right|\left.(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\right\rangle\\&\left\langle(n_p,0)_b,l_b,m_b\right|\left.L_m^{(1)}\right|\left.(n_{p1},0)_{b1},l_{b1},m_{b1}\right\rangle x_{Ka\chi a}x_{Ka1\chi a1}\;\delta_{\lambda_a\lambda_{a1}}\delta_{u_au_{a1}}\delta_{KaKa1}\delta_{l_al_{a1}}\delta_{mama1}\end{aligned} \tag{III-105}$$

We can define the matrix elements of $L_m^{(1)}$ [28]:

$$\begin{aligned}&\left\langle(n_p,0)_b,l_b,m_b\right|\left.L_m^{(1)}\right|\left.(n_{p1},0)_{b1},l_{b1},m_{b1}\right\rangle\\&\qquad=\left\langle l_{b1},m_{b1};1,m\right|\left.l_b,m_b\right\rangle(2l_b+1)^{-1/2}\left\langle(n_p,0),l_b\right\|\left.L^{(1)}\right\|\left.(n_{p1},0),l_{b1}\right\rangle\end{aligned} \tag{III-106}$$

The reduced matrix elements of $L_m^{(1)}$ are:

$$\left\langle (n_p,0), l_b \left\| L^{(1)} \right\| (n_{p1},0), l_{b1} \right\rangle = \sqrt{l_b(l_b+1)}\delta_{l_b l_{b1}}\delta_{n_p n_{p1}} \tag{III-107}$$

Then [28]

$$\left\langle (n_p,0)_b, l_b, m_b \left| L_m^{(1)} \right| (n_{p1},0)_{b1}, l_{b1}, m_{b1} \right\rangle = \left\langle l_{b1}, m_{b1}; 1, m \middle| l_b, m_b \right\rangle (2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}\delta_{l_b l_{b1}}\delta_{n_p n_{p1}} \tag{III-108}$$

Thus, with replace the Eq. (III-108) in Eq.(III-105) the matrix element of $L_m^{(1)}$ has the following form:

$$\begin{aligned}\left\langle \varphi \left| L_m^{(1)} \right| \varphi_1 \right\rangle = \sum_{\substack{K_a,\chi_{a1},\chi_a,l_a,m_a \\ l_b,m_b}} & \left\langle (\lambda,\mu)_a, \chi_a, l_a, m_a; (n_p,0)_b, l_b, m_b \middle| (\lambda,\mu); \chi, L, M_L \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_a, l_a, m_{a1}; (n_p,0)_b, l_b, m_{b1} \middle| (\lambda_1,\mu_1); \chi_1, L_1, M_{L1} \right\rangle \\ & \left\langle l_b, m_{b1}; 1, m \middle| l_b, m_b \right\rangle (2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}x_{Ka\chi a}x_{Ka\chi a1}\end{aligned} \tag{III-109}$$

By using the definition of reduce matrix elements of $Q_m^{(2)}$

$$\begin{aligned}&\left\langle (n_p,0)_b, l_b, m_b \left| Q_m^{(2)} \right| (n_{p1},0)_{b1}, l_{b1}, m_{b1} \right\rangle = \\ &\qquad \left\langle l_{b1}, m_{b1}; 2, m \middle| l_b, m_b \right\rangle (2l_b+1)^{-1/2} \left\langle (n_p,0), l_b \left\| Q^{(2)} \right\| (n_{p1},0), l_{b1} \right\rangle\end{aligned} \tag{III-110}$$

and the following relation

$$\left\langle (n_p,0), l_b \left\| Q^{(2)} \right\| (n_{p1},0), l_{b1} \right\rangle = \begin{cases} (n_p+3)\left[\dfrac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]\delta_{l_{b1}l_b}\delta_{n_p n_{p1}} \\ \left[\dfrac{(n_p-l_b)(n_p+l_b+3)(l_b+1)(l_b+2)}{(2l_b+3)(2l_b+5)}\right]\delta_{l_{b1}l_b-2}\delta_{n_p n_{p1}} \end{cases} \tag{III-111}$$

We see that two case are possible:

a)-For $l_{b1} = l_b$ -2 we can't find the square mean of the operators Ω

b)-For $l_{b1} = l_b$

In the same way we can calculate the matrix elements of $Q_m^{(2)}$

$$\begin{aligned}\left\langle \varphi \left| Q_m^{(2)} \right| \varphi_1 \right\rangle = & \sum_{\substack{K_a, K_{a1}, \chi_a, l_a, m_a \\ l_b, m_b \\ \chi_{a1}, l_{a1}, m_{a1} \\ l_{b1}, m_{b1}}} \left\langle (\lambda,\mu)_a, \chi_a, l_a, m_a; (n_p,0)_b, l_b, m_b \left| (\lambda,\mu); \chi, L, M_L \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_{a1}, l_a, m_{a1}; (n_p,0)_{b1}, l_b, m_{b1} \left| (\lambda_1,\mu_1); \chi_1, L_1, M_{L1} \right. \right\rangle \\ & \left\langle l_b, m_b; 2, m \left| l_b, m_b \right. \right\rangle (2l_b+1)^{-1/2} (n_p+3) \left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)} \right] \\ & x_{Ka\chi a} x_{Ka\chi a1}\end{aligned} \tag{III-112}$$

Using Eq.(III-112) and Eq.(III-109)

$$\begin{aligned}\left\langle \varphi \left| \Omega \right| \varphi \right\rangle = -3\sqrt{\frac{5}{2}} \sum_{\substack{um, K_a, \chi_{a1}, \chi_a, l_a, m_a \\ l_b, m_b}} & \left\langle 1u1-u \left| 00 \right. \right\rangle \left\langle 1m2u-m \left| 1u \right. \right\rangle \left\langle (\lambda,\mu)_a, \chi_a, l_a, m_a; (n_p,0)_b, l_b, m_b \left| (\lambda,\mu); \chi, L, M_L \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_{a1}, l_a, m_a; (n_p,0)_b, l_b, , m_{b1} \left| (\lambda_1,\mu_1); \chi_1, L_1, M_{L1} \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_{a1}, l_a, m_a; (n_p,0)_b, l_b, , m_{b1} \left| (\lambda_1,\mu_1); \chi_1, L_1, M_{L1} \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_{a2}, l_a, m_a; (n_p,0)_b, l_b, , m_{b2} \left| (\lambda_2,\mu_2); \chi_2, L_2, M_{L2} \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_a, l_a, m_a; (n_p,0)_b, l_b, m_b \left| (\lambda,\mu); \chi, L, M_L \right. \right\rangle \\ & \left\langle (\lambda,\mu)_a, \chi_{a2}, l_a, m_a; (n_p,0)_b, l_b, m_{b2} \left| (\lambda_2,\mu_2); \chi_2, L_2, M_{L2} \right. \right\rangle \\ & \left\langle l_b, m_{b1}; 1, m \left| l_b, m_b \right. \right\rangle (2l_b+1)^{-1/2} \sqrt{l_b(l_b+1)} \\ & \left\langle l_b, m_{b2}; 2, u-m \left| l_b, m_{b1} \right. \right\rangle (2l_b+1)^{-1/2} (n_p+3) \left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)} \right] \\ & \left\langle l_b, m_b; 1, -u \left| l_b, m_{b2} \right. \right\rangle (2l_b+1)^{-1/2} \sqrt{l_b(l_b+1)} x_{Ka\chi a} x_{Ka\chi a1} x_{Ka\chi a2} x_{Ka\chi a1} x_{Ka\chi a} x_{Ka\chi a2}\end{aligned} \tag{III-113}$$

By using the definition

$$\left\langle \varphi_1 \left| Q_m^{(2)} \right| \varphi_2 \right\rangle = \left\langle L_2 M_2 \, 2u-m \left| L_1 M_1 \right. \right\rangle (2L_1+1)^{-1/2} \left\langle \varphi_1 \left\| Q^{(2)} \right\| \varphi_2 \right\rangle \tag{III-114}$$

we can deduce

$$L_1 = L-1, L, L+1 \tag{III-115-a}$$

$$L_2 = L-1, L, L+1 \tag{III-115-b}$$

ii)The effect of Λ operator

In second we calculated the effect of Λ operator
This operator is defined as:

$$\begin{aligned}\Lambda &= \left[L \times Q \times Q \times L\right]_0^{(0)} \\ &= \sum_{kmu1u} \langle kmk-m|00\rangle \langle 1u2m-u|km\rangle \langle 2-m-u_1 1u_1|k-m\rangle L_u Q_{m-u} Q_{-m-u1} L_{u1}\end{aligned} \tag{III-116}$$

Using the definition of the coupled operators we calculated

$$\begin{aligned}&\langle \varphi|\Lambda|\varphi\rangle = \\ &\sum_{kmu1u} \langle kmk-m|00\rangle \langle 1u2m-u|km\rangle \langle 2-m-u_1 1u_1|k-m\rangle \langle \varphi|L_u Q_{m-u} Q_{-m-u1} L_{u1}|\varphi\rangle\end{aligned} \tag{III-117}$$

By using the expression of the matrix elements of L_u Eq. (III-109) and Q_m we can get this relation :

$$\langle\varphi|\Lambda|\varphi\rangle =$$

$$\sum_{\substack{kmu1u \\ K_a,\chi_{a1},\chi_a,l_a,m_a \\ l_b,m_b}} \langle kmk-m|00\rangle\langle 1u2m-u|km\rangle\langle 2-m-u_1 1u_1|k-m\rangle$$

$$\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$

$$\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a3},l_a,m_a;(n_p,0)_b,l_b,m_{b3}|(\lambda_3,\mu_3);\chi_3,L_3,M_{L3}\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$

$$\langle(\lambda,\mu)_a,\chi_{a3},l_a,m_a;(n_p,0)_b,l_b,m_{b3}|(\lambda_3,\mu_3);\chi_3,L_3,M_{L3}\rangle$$

$$\langle l_b,m_{b1};1,u|l_b,m_b\rangle\langle l_b,m_{b2};2,m-u|l_b,m_{b1}\rangle\langle l_b,m_{b3};2,-m-u1|l_b,m_{b2}\rangle\langle l_b,m_b;1,u1|l_b,m_{b3}\rangle$$

$$(2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}(2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}$$

$$(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]\ (2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]$$

$$x_{Ka\chi a}x_{Ka1\chi a1}x_{Ka2\chi a2}x_{Ka1\chi a1}x_{Ka2\chi a2}x_{Ka3\chi a3}x_{Ka\chi a}x_{Ka3\chi a3}$$

(III-118)

where

$$L_1 = L-1, L, L+1 \qquad \text{(III-119-a)}$$

$$L_2 = L_1-2, L_1-1, L_1, L_1+1, L_1+2 \qquad \text{(III-119-b)}$$

$$L_3 = L-1, L, L+1 \qquad \text{(III-119-c)}$$

iii)The effect of C_3 operator

The operator C_3 is obtained as

$$\begin{aligned} C_3 = C_3(SU(3)) &= \frac{-4}{9}\left[Q\times Q\times Q\right]_0^{(0)} - \frac{\sqrt{15}}{2}\left[L\times Q\times L\right]_0^{(0)} \\ &= \frac{-4}{9}\sum_{mu}\ \left\langle 2u2m-u\middle|2m\right\rangle\left\langle 2m2-m\middle|00\right\rangle Q_u Q_{m-u} Q_{-m} \\ &\quad -\frac{\sqrt{15}}{2}\sum_{mu}\left\langle 1u1-u\middle|00\right\rangle\left\langle 1m1u-m\middle|1u\right\rangle L_m Q_{u-m} L_{-u} \end{aligned} \tag{III-120}$$

Then:

$$\begin{aligned} C_3 = \frac{-4}{9}\frac{(-1)^m}{\sqrt{5}}\sum_{mu}\ &\left\langle 2u2m-u\middle|2m\right\rangle Q_u Q_{m-u} Q_{-m} \\ +\frac{\sqrt{15}}{2}\frac{(-1)^u}{\sqrt{3}}\sum_{mu}\ &\left\langle 1m2u-m\middle|1u\right\rangle L_m Q_{u-m} L_{-u} \end{aligned} \tag{III-121}$$

a) We take the first part of the C_3 operator in Eq. (III-121) which is defined as:

$$A = \sum_{mu}\ \left\langle 2u2m-u\middle|2m\right\rangle Q_u Q_{m-u} Q_{-m} \tag{III-122}$$

We have:

$$\begin{aligned} \left\langle\varphi\middle|A\middle|\varphi\right\rangle &= \sum_{mu}\ \left\langle 2u2m-u\middle|2m\right\rangle\left\langle\varphi\middle|Q_u Q_{m-u} Q_{-m}\middle|\varphi\right\rangle \\ &= \sum_{mu\varphi 1\varphi 2}\ \left\langle 2u2m-u\middle|2m\right\rangle\left\langle\varphi\middle|Q_u\middle|\varphi_1\right\rangle\left\langle\varphi_1\middle|Q_{m-u}\middle|\varphi_2\right\rangle\left\langle\varphi_2\middle|Q_{-m}\middle|\varphi\right\rangle \end{aligned} \tag{III-123}$$

In the same way we can obtain this following form:

$$\langle\varphi|A|\varphi\rangle=\frac{-4}{9}\frac{(-1)^m}{\sqrt{5}}\sum_{\substack{m,u\\ \chi_a,l_a,m_a\\ l_b,m_b}}\langle 2u2m-u|2m\rangle\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle l_b,m_{b1};2,m|l_b,m_b\rangle\langle l_b,m_{b2};2,m-u|l_b,m_{b1}\rangle\langle l_b,m_b;2,-m|l_b,m_{b2}\rangle$$
$$(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]\quad(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]$$
$$(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]x_{Ka\chi a}x_{Ka\chi a1}x_{Ka\chi a2}x_{Ka1\chi a1}x_{Ka\chi a}x_{Ka\chi a2}$$

(III-124)

where

$$L_2=L-2,L-1,L,L+1,L+2 \quad \text{(III-125-a)}$$
$$L_1=L-2,L-1,L,L+1,L+2 \quad \text{(III-125-b)}$$

b) We take the second part of the C_3 operator (III-121) which is defined as:

$$B=-\frac{\sqrt{15}}{2}\left[L\times Q\times L\right]_0^{(0)}=\frac{\sqrt{3}}{3}\Omega \quad \text{(III-126)}$$

Thus, the matrix elements of B is :

$$\langle\varphi|B|\varphi\rangle=\frac{\sqrt{3}}{3}\langle\varphi|\Omega|\varphi\rangle \quad \text{(III-127)}$$

Where

$$L_1=L-1,L,L+1 \quad \text{(III-128-a)}$$
$$L_2=L-1,L,L+1 \quad \text{(III-128-b)}$$

From Eq (III-124) and (III-127) we can write the mean square of C_3 operator as

$$\langle\varphi|C_3|\varphi\rangle=\frac{-4}{9}\frac{(-1)^m}{\sqrt{5}}\sum_{\substack{m,u\\ \chi_a,l_a,m_a\\ l_b,m_b}}\langle 2u2m-u|2m\rangle\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle l_b,m_{b1};2,m|l_b,m_b\rangle\langle l_b,m_{b2};2,m-u|l_b,m_{b1}\rangle\langle l_b,m_b;2,-m|l_b,m_{b2}\rangle$$
$$(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]\quad(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]$$
$$(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]x_{Ka\chi a}x_{Ka\chi a1}x_{Ka\chi a2}x_{Ka1\chi a1}x_{Ka\chi a}x_{Ka\chi a2}$$

$$-\frac{\sqrt{15}}{2}\sum_{\substack{um,K_a,\chi_{a1},\chi_a,l_a,m_a\\ l_b,m_b}}\langle 1u1-u|00\rangle\langle 1m2u-m|1u\rangle\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a1},l_a,m_a;(n_p,0)_b,l_b,,m_{b1}|(\lambda_1,\mu_1);\chi_1,L_1,M_{L1}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle(\lambda,\mu)_a,\chi_a,l_a,m_a;(n_p,0)_b,l_b,m_b|(\lambda,\mu);\chi,L,M_L\rangle$$
$$\langle(\lambda,\mu)_a,\chi_{a2},l_a,m_a;(n_p,0)_b,l_b,m_{b2}|(\lambda_2,\mu_2);\chi_2,L_2,M_{L2}\rangle$$
$$\langle l_b,m_{b1};1,m|l_b,m_b\rangle(2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}$$
$$\langle l_b,m_{b2};2,u-m|l_b,m_{b1}\rangle(2l_b+1)^{-1/2}(n_p+3)\left[\frac{l_b(l_b+1)}{6(2l_b-1)(2l_b+3)}\right]$$
$$\langle l_b,m_b;1,-u|l_b,m_{b2}\rangle(2l_b+1)^{-1/2}\sqrt{l_b(l_b+1)}x_{Ka\chi a}x_{Ka\chi a1}x_{Ka\chi a2}x_{Ka\chi a1}x_{Ka\chi a}x_{Ka\chi a2}$$

(III-129)

Thus, by Replacing Eq.(III-113), Eq.(III-118) and Eq.(III-129) in (III-91) we will find the expression of $\langle\varphi|H'|\varphi\rangle$ and E'. We have developed the Hamiltonian of the SU(3) limits by introduction the higher order terms. One can use the numerical calculations using the values of Clebsch-Gordan coefficients of so(3) and the Coupling coefficients of su(3) and overlap integral for finding a more accurate results .

III-8-2.Higher order terms in the transition operators

One can introduce two-body terms in the transition operators [27]

$$T_2^{(L)} = \sum_{\alpha\beta} c_{\alpha\beta\gamma\delta}^{(L)} b_\alpha^+ b_\beta^+ b_\gamma b_\delta \tag{III-130}$$

They can be written explicitly as

$$T_{2,u}^{(L)} = \sum_{\substack{k,i,i' \\ k',j,j'}} t_{k,i,i',k',j,j'}^{(L)} \left[[b_{i'}^+ \times b_i^+]^{(k)} \times [\tilde{b}_j \times \tilde{b}_{j'}]^{(k')} + [b_{j'}^+ \times b_j^+]^{(k')} \times [\tilde{b}_{i'} \times \tilde{b}_i]^{(k)} \right]_u^{(L)} \tag{III-131}$$

In the nuclear vibron model we have derived the two-body terms in the transition operators including only s' and p bosons. The obtained results can be written, using the recoupling formula, as follows (including the one-body terms).

$$\begin{aligned} T_{2,u}^{(E_0)} &= a_0 [s^+ \times \tilde{s}]_0^{(0)} + b_0 [d^+ \times \tilde{d}]_0^{(0)} + c_0 [s'^+ \times \tilde{s}']_0^{(0)} + d_0 [p^+ \times \tilde{p}]_0^{(0)} \\ &+ e_0 [s'^+ \times s'^+]_0^{(0)} \times [\tilde{s}' \times \tilde{s}]_0^{(0)} + f_0 \left[[p^+ \times p^+]^{(0)} \times [\tilde{p} \times \tilde{p}]^{(0)} \right]_0^{(0)} \\ &+ g_0 \left[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{p}]^{(2)} \right]_0^{(0)} \\ &+ h_0 \left[[s'^+ \times s'^+]^{(0)} \times [\tilde{p} \times \tilde{p}]^{(0)} + [p^+ \times p^+]^{(0)} \times [\tilde{s}' \times \tilde{s}']^{(0)} \right]_0^{(0)} \end{aligned} \tag{III-132-a}$$

$$\begin{aligned} T_{2,u}^{(E_1)} &= a_1^- \left[p^+ \times \tilde{s}' + s'^+ \times \tilde{p} \right]_u^{(1)} \\ &+ b_1^- \left[[p^+ \times p^+]^{(0)} \times [\tilde{p} \times \tilde{s}']^{(1)} + [p^+ \times \tilde{s}']^{(0)} \times [\tilde{p} \times \tilde{p}]^{(0)} \right]_u^{(1)} \\ &+ c_1^- \left[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{s}']^{(1)} + [p^+ \times \tilde{s}']^{(1)} \times [\tilde{p} \times \tilde{p}]^{(2)} \right]_u^{(1)} \\ &+ d_1^- \left[[s'^+ \times s'^+]^{(0)} \times [\tilde{p} \times \tilde{s}']^{(1)} + [p^+ \times \tilde{s}']^{(1)} \times [\tilde{s}' \times \tilde{s}']^{(0)} \right]_u^{(1)} \end{aligned} \tag{III-132-b}$$

$$\begin{aligned} T_{2,u}^{(M_1)} &= a_1^+ [\tilde{d} \times d^+]_u^{(1)} + b_1^+ [p^+ \times \tilde{p}]_u^{(1)} \\ &+ c_1^+ \left[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{p}]^{(2)} \right]_u^{(1)} \end{aligned} \tag{III-132-c}$$

$$
\begin{aligned}
T^{(E_2)}_{2,u} = a_2 \Big[d^+ \times \tilde{s} + s^+ \times \tilde{d} \Big]^{(2)}_u + b_2 [d^+ \times \tilde{d}]^{(2)}_u + c_2 [p^+ \times \tilde{p}]^{(2)}_u \\
+ d_2 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{s}' \times \tilde{s}']^{(0)} + [s'^+ \times s'^+]^{(0)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(2)}_u \\
+ l_2 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(2)}_u \\
+ f_2 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{s}' \times \tilde{s}']^{(0)} + [s'^+ \times s'^+]^{(0)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(2)}_u
\end{aligned}
\qquad \text{(III-132-d)}
$$

$$
\begin{aligned}
T^{(M_3)}_{2,u} = a_3 [d^+ \times \tilde{d}]^{(3)}_u \\
+ b_3 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(3)}_u
\end{aligned}
\qquad \text{(III-132-e)}
$$

$$
T^{(E_3)}_{2,u} = c_3 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{s}']^{(1)} + [p^+ \times s'^+]^{(1)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(3)}_u
\qquad \text{(III-132-f)}
$$

$$
\begin{aligned}
T^{(E_4)}_{2,u} = a_4 [d^+ \times \tilde{d}]^{(4)}_u \\
+ b_4 \Big[[p^+ \times p^+]^{(2)} \times [\tilde{p} \times \tilde{p}]^{(2)} \Big]^{(4)}_u
\end{aligned}
\qquad \text{(III-132-g)}
$$

The matrix elements are straight forward to calculate and depend only on the quantum numbers $(\lambda,\mu),\chi$ and L of the combined SU(3) and O(3) groups. The higher order terms introduced in the $B(En/Mn)$ could also lead to better results in the reduced transition probabilities.

General conclusion

Our work study the cluster model in Er isotopes for explanation the ground state bands and transition electromagnetic in the tow aspects the first is semi-classical aspect and the second is algebraic aspect, we stared with semi-classical aspect using cluster-core potential and we applied a principle of maximum stability to determine the most favoured core-cluster decomposition of Er isotopes, using the fixed parameter values for all case. We have provided a consistent cluster model description of the ground bands in Er isotopes. We have examined 7 even-even isotopes $^{158-170}$Er using a following expressions for the cluster core potential parameters: $V0$ = 54.0 MeV, a = 0.73 fm and x = 0.33 and global quantum number G = $4A_2$ and chosen this values of the parameters it entails to give a simultaneously optimized description of the energies of the low-lying positive parity states of Er isotopes , Comparisons of cluster-core binding energies with the corresponding liquid drop values and the penetrabilities, for breakup into possible cluster-core pairs, The potential radius is fitted to the experimental value for ground state energy. Energies and wave functions are then obtained by numerical solution of the Schrödinger equation using harmonic oscillation base.

This approach has been shown to give a good account of the energies and $E2$ electromagnetic transitions of the lowest $J^{\pi} = 0+, 2+, 4+ \ldots G^{+}$ bands of states. Excitation energies and E2 transition strengths (or quadrupole moments) are reproduced in good agreement with experiment. The results suggest the mode of clusterization associated with the ground state band. The calculated spectra are in reasonable agreement with the available data.

We presented algebraic aspect This approach is based on an algebraic treatment of both deformation and clustering described by the group U(6), and the group U(4) respectively, the group U(6), that is describe the collective quadrupole excitations and by group U(4) which describe the dipole degree of freedom. The U(6) algebra admits three dynamic symmetries, described by and U(4) algebra admits two dynamic symmetries, described by the group chains. the resultant model, U(6) $\otimes$ U(4), is suited for applications to a large variety of nuclei.

We discuss a particular limit of this model, SU(3), this limit corresponds to an harmonic vibrations of the cluster in an axially deformed nucleus. We can solved this limit by using analytically methods and derive analytic formulas for energies, and electromagnetic transition. The analytic expressions for energy levels have been reproduced in good agreement with experiment by several authors that clarify the role of clustering in nuclei. In the finals we study the effects of the high order in the Hamiltonian of SU(3) limit and electromagnetic transition .

Appendix A

The eigenfunctions of the tridimensional isotropic harmonic oscillator

The Hamiltonian for the three-dimensional, spherically symmetric harmonic oscillator is of the form

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}r^2 - U_0 \tag{A-1}$$

We solve the stationary Schrödinger equation

$$H\psi_{nl} = E_{nl}\psi_{nl} \tag{A-2}$$

(i) Using a Cartesian basis we have

$$U(r) = -U_0 + \frac{1}{2m}\omega^2(x^2 + y^2 + z^2) \tag{A-3}$$

The three (x, y,z) specific one-dimensional oscillator eigenvalue equations become

$$\left[\frac{d^2}{dx^2} + \frac{2m}{\hbar^2}\left(E_1 + \frac{U_0}{3} - \frac{1}{2}m\omega^2x^2\right)\right]\varphi_1(x) = 0 \tag{A-4-a}$$

$$\left[\frac{d^2}{dy^2} + \frac{2m}{\hbar^2}\left(E_2 + \frac{U_0}{3} - \frac{1}{2}m\omega y^2\right)\right]\varphi_1(y) = 0 \tag{A-4-b}$$

$$\left[\frac{d^2}{dz^2} + \frac{2m}{\hbar^2}\left(E_3 + \frac{U_0}{3} - \frac{1}{2}m\omega^2z^2\right)\right]\varphi_1(e) = 0 \tag{A-4-c}$$

with

$$E = E_1 + E_2 + E_3 \tag{A-5}$$

The three eigenvalues are then

$$E_1 = \hbar\omega(n_1 + \frac{1}{2}) - \frac{U_0}{3} \tag{A-6-a}$$

$$E_2 = \hbar\omega(n_2 + \frac{1}{2}) - \frac{U_0}{3} \tag{A-6-b}$$

$$E_3 = \hbar\omega(n_3 + \frac{1}{2}) - \frac{U_0}{3} \tag{A-6-c}$$

or

$$E = \hbar\omega(N + \frac{3}{2}) - U_0 \tag{A-7}$$

($N = n_1 + n_2 + n_3$, with n_1,n_2,n_3 three positive integer numbers 0,1,…)

The wavefunctions for the one-dimensional oscillator are the Hermite polynomials, characterized by the radial quantum number n_1, so

$$\varphi_1(x) = N_1 \exp\left(\frac{-m\omega}{\hbar^2}x^2\right) H_{n_1}(\nu x) \qquad \left(\nu = \sqrt{\frac{m\omega}{\hbar}}\right) \tag{A-8}$$

$$\varphi(x, y, z) = \varphi_1(x)\varphi_2(y)\varphi_2(z) \tag{A-9}$$

N_1 N_2 N_3 are normalization coefficients.

(ii) In spherical coordinates, n, l are quantum numbers characterizing the eigenfunctions and will have to be further specified. The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2} \tag{A-10}$$

where L is the angular-momentum operator. The eigenfunctions of L^2 are the spherical harmonics

$$L^2 y_{lm}(\vartheta, \varphi) = \hbar^2 l(l+1) y_{lm}(\vartheta, \varphi) \tag{A-11}$$

To separate the angular and radial parts of the wave function u_{nlm} we try writing

$$u_{nlm}(r, \vartheta, \varphi) = \frac{R_{nl}(r)}{r} y_{lm}(\vartheta, \varphi) \tag{A-12}$$

Differential equation for the radial part of the wave function $R_{nl}(r)$ is given by (after dropping U_0)

$$\left[\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E_{nl} + -\frac{1}{2}m\omega^2 r^2 - \frac{l(l+1)}{r^2}\right)\right] R_{nl}(r) = 0 \tag{A-13}$$

For the harmonic oscillator in spherical coordinates the complete wave functions should be familiar from elementary quantum mechanics; they are given by

$$u_{nlm}(r, \vartheta, \varphi) = \sqrt{\frac{2^{n+l+2}}{n!(2n+2l+1)\sqrt{\pi}\, x_0}} \frac{r^l}{x_0{}^l} L_n^{l+1/2}\left(\frac{r^2}{x_0{}^2}\right) e^{\left(\frac{-r^2}{2x_0^2}\right)} y_{lm}(\Omega) \tag{A-14}$$

with

$$x_0 = \sqrt{\frac{\hbar}{m\omega}} \tag{A-15}$$

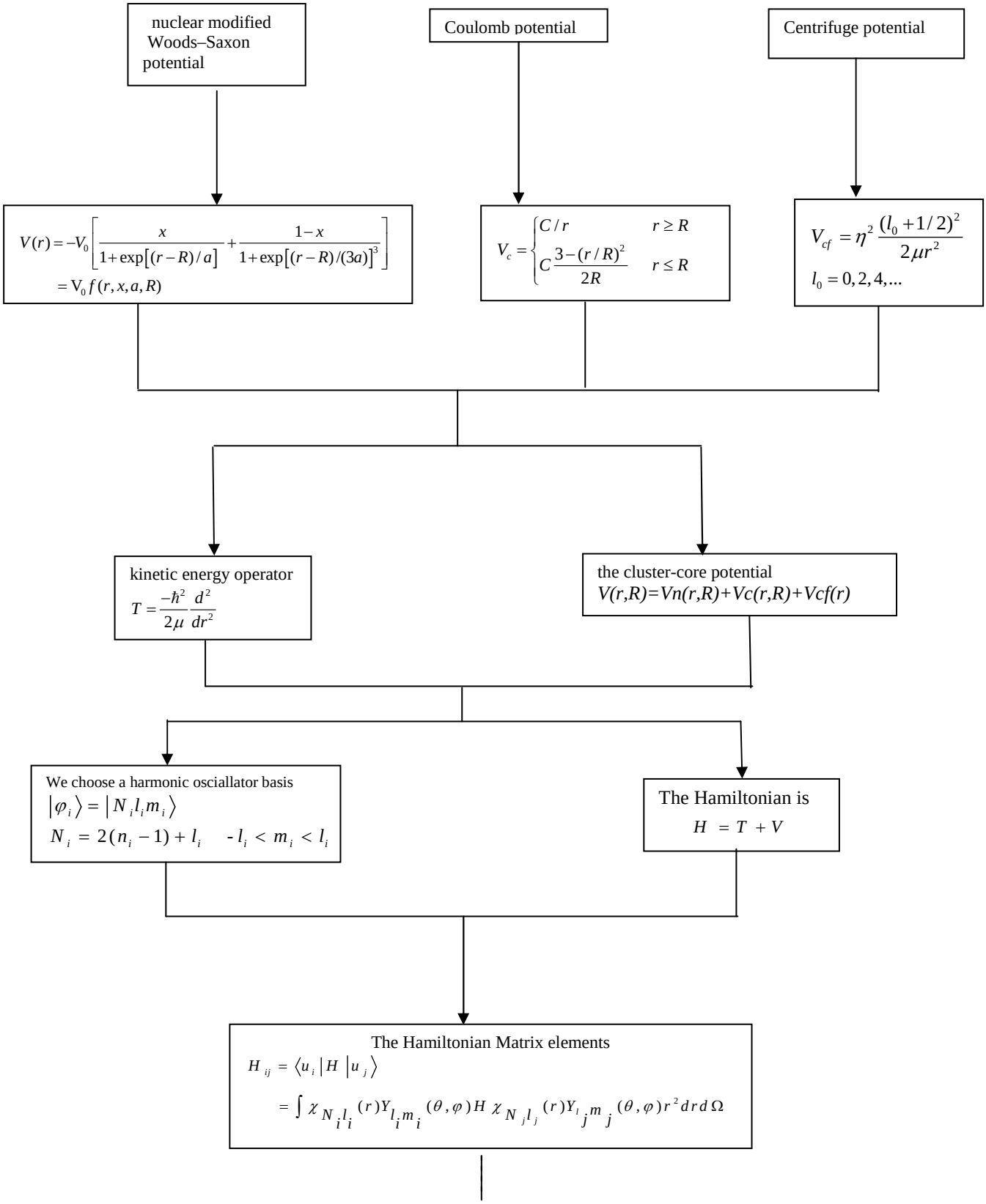
The symbol $L_n^l(x)$ stands for the generalized Laguerre polynomial.
The eigenenergies (A-7) are determined by the principal quantum number $N=2(n-1)+l$ and are $\frac{1}{2}(N+1)(N+2)$ fold degenerate (see table A-1).

N=2n+l	n	l	Energy	Number of the degenerate states	
0	0	0(s)	$\frac{3}{2}\hbar\omega$	1	
1	0	1(p)	$\frac{5}{2}\hbar\omega$	3	
2	1	0(s)	$\frac{7}{2}\hbar\omega$	1	6
	0	2(d)		5	
3	1	1(3)	$\frac{9}{2}\hbar\omega$	3	10
	0	3(f)		7	
4	2	0(s)		1	16
	1	2(d)	$\frac{11}{2}\hbar\omega$	5	
	0	4(g)		9	

Table A-1 Spectrum of the three-dimensional harmonic oscillator

Appendix B

Algorithm of numerical calculations of the energy levels.



Example: for N=2 we have this matrix with dimension 10×10

$$H = \begin{array}{c} \begin{matrix} |000\rangle & |111\rangle & |110\rangle & |11{-}1\rangle & |200\rangle & |222\rangle & |221\rangle & |220\rangle & |22{-}1\rangle & |22{-}2\rangle \end{matrix} \\ \begin{bmatrix} H_{11} & 0 & 0 & 0 & H_{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & H_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_{55} & 0 & 0 & 0 & 0 & 0 \\ H_{51} & 0 & 0 & 0 & 0 & H_{66} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_{77} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{88} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{99} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_{10} \end{bmatrix} \end{array}$$

Diagonalization this matrix using Jacobi subroutine

Obtain the eigenvalues $e_1(L), e_1(L), \ldots, e_{SN}(L)$ of H matrix that are classified according to L

We take $e_{min}(L) = \min(e_1(L), e_1(L), \ldots, e_{SN}(L))$

Energy Levels of ground stat bound

$E_2^+ = e_{min}(L=2) - e_{min}(L=0)$

$E_4^+ = e_{min}(L=4) - e_{min}(L=0)$

$E_6^+ = e_{min}(L=6) - e_{min}(L=0)$

$E_8^+ = e_{min}(L=8) - e_{min}(L=0)$

Appendix C

Lie algebras

C-1.Definition

A Lie algebra *L* is a vector space of elements a, *b, c,* ...that satisfy the following condition

a) the commutator of two elements is again an element of the algebra

$$[a,b] \in L \quad \text{for } a,b \in L \qquad (C\text{-}1)$$

b) a linear combination of two elements with the numbers is again an element of the algebra

$$[\alpha \cdot a + \beta.b, c] = \alpha \cdot [a,c] + \beta.[b,c] \text{ for } a,b,c \in L \qquad (C\text{-}2)$$

c) interchanging both elements of a commutator results in

$$[a,b] = -[b,a] \text{ for } a,b \in L \qquad (C\text{-}3)$$

d) Jacobi identity has to be satisfied as follows

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0 \text{ for } a,b,c \in L \qquad (C\text{-}4)$$

The commutator of *a* and *b* is defined as

$$[a,b] = a.b - b.a \text{ for } a,b \in L \qquad (C\text{-}5)$$

Lie algebras can be represented by matrices with certain symmetry properties or by operators. If the algebra constitutes a n-dimensional vector space (a_1 ,$a_2 \ldots a_n$). The commutator of any two elements is a linear combination of the elements in the Lie algebra

$$\left[a_i, a_j\right] = \sum_{ij} c_{ij}^k a_k \qquad (C\text{-}6)$$

D-2.Lie Groups

Associated with each Lie algebra there is a Lie group. A set of elements *A,B, C* . . ., forms a group *G* if it satisfies the following axioms:

1)identity property

$$AI = IA = A \qquad (C\text{-}7)$$

2) closure property.

$$AB = C \tag{C-8}$$

3) Inverse property .

$$A^{-1}A = AA^{-1} = I \tag{C-9}$$

4) Associatively property.

$$A(BC) = (AB)C \tag{C-10}$$

For example, the inverse of a rotation about some axis through an angle φis a rotation about the same axis through an angle –φ.
The groups of interest to us can divided into two different typs :
Continuous groups whose continuous parameters such as the translation and rotation groups.
Discrete groups that are consist of discrete operations.
The application of a group elements $g(\alpha)$ on a vector $\boldsymbol{x}=(x_1,x_2,\ldots,x_N)$ in an N-dimensional space can be written as

$$\boldsymbol{x}'=g(\alpha)x=f(x,\alpha) \tag{C-11}$$

the generators of the group *G* is define as following

$$G_i = \sum_{k=1}^{N} U_i^k \frac{\partial}{\partial x_k} \tag{C-12}$$

Where

$$U_i^k = \frac{\partial}{\partial \alpha_i} f(x,\alpha)\Big|_{\alpha=0} \tag{C-13}$$

D-3.Casimir Operators
An operator which commutes with all the elements of a Lie algebra, *g*, is called an invariant, or Casimir operator, *C*

$$[C,X\rho] = 0 \text{ for any } X\rho \in g \tag{C-14}$$

The operator is called of order *p*, if it is built from products of *p* elements

$$C_p = \sum_{\alpha_1,\alpha_2,\alpha_3\ldots\alpha_n} f^{\alpha_1,\alpha_2,\alpha_3\ldots\alpha_n} a_{\alpha_1}.a_{\alpha_2}\ldots a_{\alpha_n} \tag{C-15}$$

The first order casimir invariant of the U(n) algebras is given by

$$C_1\left[U(n)\right] = \sum_{i=1}^{n} G_i^i \tag{C-16}$$

And

$$\left[C_2\left[U(n)\right], G_i^i\right] = 0 \qquad \text{(C-17)}$$

We may choose the basis a way that the t polynomial $P_{\rho k}\,(\,b_{\rho k}{}^{+})$ are eigenfunctions of G_i^i

$$G_i^i\;\; P_{\rho k}\,(\,b_{\rho k}{}^{+})\left|0\right\rangle = \omega_i\, P_{\rho k}\,(\,b_{\rho k}{}^{+})\left|0\right\rangle \qquad \text{(C-18)}$$

The set of numbers ($\omega_{1,}\,\omega_{2,\ldots}\,,\,\omega_n$) is called the weight of the polynomial $P_{\rho k}\,(\,b_{\rho k}{}^{+})$.
We define highest-weight polynomial by the following relation

$$\begin{aligned} &G_i^i\;\; P_{\rho k}\,(\,b_{\rho k}{}^{+})\left|0\right\rangle = h_i\, P_{\rho k}\,(\,b_{\rho k}{}^{+})\left|0\right\rangle \qquad \text{for } i=1,2,\ldots,n \\ &G_i^i\;\; P_{\rho k}\,(\,b_{\rho k}{}^{+})\left|0\right\rangle = 0 \qquad\quad i \prec j \quad i,j=1,2,\ldots,n \end{aligned} \qquad \text{(C-19)}$$

The degree of the polynomial $P_{\rho k}\,(\,b_{\rho k}{}^{+})$ is

$$h_1 + h_2 + \ldots + h_n = N \qquad \text{(C-20)}$$

where

$$h_1 \geq h_2 \geq \ldots \geq 0 \qquad \text{(C-21)}$$

The irreducible representations of U(n) algebra are characterized by a Young partition $\left[h_1, h_2, \ldots, h_n\right]$ of the integer N
the eigenvalue for C_2[U(n)] is define as

$$C_2\left[U(n)\right] P_{h\omega}(b_{\rho k}^{+})\left|0\right\rangle = \left(\sum_{i=1}^{n} h_i^2 + \sum_{i \prec j=1}^{n} (h_i - h_j)\right) P_{h\omega}(b_{\rho k}^{+})\left|0\right\rangle \qquad \text{(C-22)}$$

The second order casimir invariant of the U(n) algebras is given by

$$C_2\left[U(n)\right] = \sum_{ij=1}^{n} G_i^j G_j^i \qquad \text{(C-23)}$$

Then

$$\left[C_2\left[U(n)\right], G_i^j\right] = 0 \qquad \text{(C-24)}$$

The second order casimir invariant of the $SO($n) algebras is given by

$$C_2\left[SU(n)\right] = \frac{1}{2}\sum_{ij=1}^{n} \Lambda_{ij}\Lambda_{ji} \qquad \text{(C-25)}$$

Then

$$\left[C_2\left[U(n)\right], \Lambda_{ij}\right] = 0 \qquad \text{(C-26)}$$

One can define a highest weight polynomial $P_{\rho k}$ ($b_{\rho k}^{+}$) for which

$$\Lambda_{mm}\ P_{\rho k}\left(b_{\rho k}^{+}\right)\left|0\right\rangle = \lambda_{l-m+1}\ P_{\rho k}\left(b_{\rho k}^{+}\right)\left|0\right\rangle \quad \text{for } m=l,l\text{-}1,\ldots,1$$
$$\Lambda_{mm'}\ P_{\rho k}\left(b_{\rho k}^{+}\right)\left|0\right\rangle = 0 \qquad m \succ m' \succ -m \qquad \text{(C-27)}$$

λ is the irreducible representations of *SO*(n) algebra

1)for *SO(*2l+1) the l numbers of the highest weight satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_1 \geq \lambda_{l\text{-}1} \geq \lambda_l \qquad \text{(C-28)}$$

and, the casimir invariant C_2[*SO*(2l+1)] has the eigenvalue

$$\sum_{k=1}^{l} \lambda_k \left(\lambda_k + 2l + 1 - 2k\right) \qquad \text{(C-29)}$$

2)for *SO*(2l+1) the l numbers of the highest weight satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_1 \geq \lambda_{l\text{-}1} \geq 0 \qquad \lambda_{l\text{-}1} \geq \left|\lambda_l\right|_l \qquad \text{(C-30)}$$

While, the casimir invariant C2[SO(2l+1)] has the eigenvalue

$$\sum_{k=1}^{l} \lambda_k \left(\lambda_k + 2l - 2k\right) \qquad \text{(C-31)}$$

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Résume

Le développement de la physique des noyaux atomiques a souvent été accompagnée par un échange d'idées avec d'autres domaines de la physique, en particulier avec la physique atomique, moléculaire, la physique des particules élémentaires ; hydrodynamique .Ces deux derrières disciplines ont joué un rôle important dans la physique nucléaire. La formation des clusters est un aspect fondamental de la dynamique de plusieurs corps nucléaire avec la formation de champ principal. Aspects de Cluster apparaissent en abondance dans de nombreux problèmes à la fois dans la structure nucléaire et collisions nucléaires. L'hypothèse de base de ce modèle est que les noyaux peuvent être décrits en termes d'un système se composants de deux noyaux, chacun avec ses caractéristiques de l'état libre; interagissant entre eux par un potentiel local. Nous avons traité ce modèle en utilisant les aspects suivants :

Aspect semi classique:

L'hypothèse de base de ce modèle est que ces noyaux peuvent être décrit en termes d'un système de deux composantes de noyaux. Nous considérons d'abord un noyau. (Z_T; A_T) décomposé en cœur et cluster configuration (Z_1; A_1) + (Z_2; A_2), chacune avec ses caractéristiques de l'état libre; interagir par un potentiel local. Cependant, beaucoup de décomposition binaire satisfait à cette exigence minimale. Ce choix doit se faire par référence aux énergies de liaison de cluster et de cœur. Si le parent noyau peut être divisé en cluster et le cœur qui sont à la fois doublement magique, alors ce sera la combinaison la plus favorisée. Le choix approprié de décomposition de cluster et le cœur d'un noyau donné est plus important lorsque l'application du modèle de cluster à ce noyau. Nous avons donc recherché les maxima des quantités

$$D(1,2) = \left[B_A(Z_1, A_1) - B_M(Z_1, A_1) \right] + \left[BA(Z_2, A_2) - B_M(Z_2, A_2) \right]$$

Dans les calculs actuels, pour chaque cluster les relations entre la charge ; Z_2 et la mass ; A_2 et $A_2 + 2$; qui se rapprochent le plus de satisfaire à cette contrainte de non-dipôle est:

$$\frac{Z_1}{A_1} \geq \frac{Z_T}{A_T} \geq \frac{Z_2}{A_2 + 2}$$

Les énergies et fonctions d'ondes de mouvement relatif sont obtenues par la résolution l'équation de Schrödinger

$$\frac{-\hbar^2}{2\mu}\frac{d^2\chi_{nL}}{dr^2}+\left[\frac{\hbar^2 L(L+1)}{2\mu r^2}+V_N(r)+V_C(r)\right]\chi_{nL}(r)=E_{nL}\chi_{nL}(r)$$

Les états possibles de ce système se assemblé en bandes, chaque bande indiquer par sa valeur de nombre quantique global G + L = 2N, où N est le nombre de nœuds et L le moment cinétique de l'état dans la bande. Une même valeur de G correspond donc à une bande des états de partie positive.

$$J^{\pi} = 0^+, 2^+, 4^+, \ldots, G^+ .$$

En application nous chosions les isotopes de Er. Dans ce modèle les deux cluster et cœur sont spinless. Les énergies des états fondamentaux de bande dans chacune des isotopes de Er étudier sont obtenus directement par la résolution de l'équation de Schrödinger en utilisant une Méthode numérique, par projection la fonction d'onde d'un système sur la base fonctions d`onde de l`oscillateur harmonique en utilisant le programme de Fortran. Nous avons obtenu une bonne concordance entre le résultat théorique et expérimentale.

Aspect algébrique:

Il y'a des phénomènes dans le noyau qui ne peuvent être facilement décrits par le modèle de couche ou le modèle collectif. Pour les descriptions ces phénomènes, un troisième modèle a été présenté, c'est le modèle de cluster. Ce modèle est utilisé pour étudier les noyaux comme état des structures moléculaires. Dans les applications aux noyaux lourds, l'un des deux clusters ou les deux peuvent se déformer. Ceci implique un couplage des degrés de liberté d'un dipôle avec le quadrupole. Dans cette partie, nous utilisons les algèbres U (4) et U (6) Nous pouvons traiter les α- particules par intégrer les deux en grand cluster (en intégrant les deux α dans un grand cluster), cette hypothèse conduit à réduire la structure du groupe au groupe suivant

$$G = U(6) \otimes U(4)$$

Nous discutons un traitement algébrique des systèmes de deux clusters, dans lequel le problème de la valeur propre est résolu par la diagonalisation de la matrice au lieu de résoudre un ensemble d'équations différentielles. Dans le modèle cluster algébriques (ACM), la méthode de quantification des bosonic est utilisée. Le U (6) $\otimes$ U (4) Lie algèbre, étudié par lesopérateurs.

Le Hamiltonien de l'algèbres U (6) $\otimes$ U (4). peut être écrite comme

$$H = H_a + H_b + H_{ab}$$

H_a : Hamiltonien de l'algèbre U (6) dans boson s,d étalonné pour une ou deux corps, H_b : Hamiltonien de l'algèbre U (4) dans s,p boson mesuré pour une ou deux corps interactions. Cette Hamiltoniens, peut être diagonalisé analytiquement, dans certains cas, cette travail est obtenu par introduire la notion de symétries dynamique du système couplé de (s, d) et (s ", p) bosons.

Nous avons présenté ici la α-clustering SU (3) limite, Cette limite correspond à une harmonique vibration du cluster dans une déformation axiale en noyau, nous pouvons résoudre cette limite en utilisant les méthodes analytiques.

La structure du spectre de la α-clusters sont dépend du moment angulaire des modes d'excitation élémentaire à partir des laquelle les α-clusters sont construits

Abstract

The clustering phenomenon is field of research in nuclear physics. the cluster model that reduces many bodies problem to a tow bodies one, known as microscopic cluster model, has been applied to the structure (energy levels and B(E2)) of $^{158\text{-}170}$Er isotopes. The corresponding Schrödinger equation has been solved numerically after a projection on a harmonic oscillation basis. On the other hand, an extension of the SU(3) limit of the algebraic cluster model (the nuclear vibron model) has been derived.

Resume

Le phénomène du clustering est un nouveau champ de recherche en physique nucléaire. Le modèle du clustering, un modèle qui réduit le système de plusieurs corps à celui deux corps, connu sous le nom « le modèle du clustering microscopique » a été appliqué à la structure des isotopes $^{158\text{-}170}$Er (les niveaux d'énergie et B(E2)). L'équation de Schrödinger correspondante été résolu numériquement après la projection sur la base de l'oscillateur harmonique. Par ailleurs, une extension de la limite SU(3) du modèle algébrique du clustering (le modèle du vibron nucléaire) a été dérivée.

ملخص

إن نموذج العنقود كان مجالا للأبحاث المتعلقة بالفيزياء النووية. نموذج العنقود المطبق في الفيزياء النووية يقوم أساس باختزال الجملة المتعدد الأجسام إلى جملة ذات جسمين, وهذا يستند إلى مفعول الزوجية , فهذا النموذج يدرس مستويات طاقة النواة أولا بواسطة حل معادلة شرودينغر وثانيا باستعمال النموذج IBM والنموذج Vibron model النتائج المتحصل عليه توافق إلى حد بعيد النتائج التجريبية.